CONTROL THEORY (TSRT09, TSRT06)

Exercises & solutions

3 oktober 2023

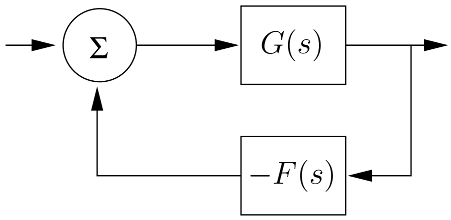
Reading instructions

- The names and numbers of the chapters in this exercise collection are consistent with the names and numbers of the chapters in the textbook.
- Starred ($*$) exercises deals with discrete-time systems and are optional.

1 Introduction

1.1

Consider the linear feedback control system given by the figure below.



Show that if the small gain theorem (Swe: lågförstärkningssatsen) is fulfilled the Nyquist criterion is also fulfilled.

Solution

1.2

Consider a static nonlinear system described by an ideal relay given by the function

$$y(t) = f(u(t)) = \left\{ \begin{array}{cc} 1, & u > 0 \\ 0, & u = 0 \\ -1, & u < 0 \end{array} \right) .$$

What is the gain of the relay?

Solution

1.3

Consider the system

$$Y(s) = G(s)U(s) \qquad G(s) = \frac{2}{s^2 + 2s + 2}$$

The control signal goes through a valve with saturation

$$\tilde{u}(t) = \begin{cases} 1, & \text{if } u(t) > 2 \\ \frac{1}{2}u(t), & \text{if } |u(t)| \leq 2 \\ -1, & \text{if } u(t) < -2 \end{cases}$$

The output is thus

$$y(t) = G(p)\tilde{u}(t).$$

The system is controlled using proportional feedback, i.e. $u(t) = -Ky(t)$. For what values of K is the closed-loop system guaranteed to be stable according to the small gain theorem?

Solution

1.4

Compute the norms $\|\cdot\|_\infty$ and $\|\cdot\|_2$ of the continuous-time signals

(a)

$$y(t) = \begin{cases} a\sin(t), & t > 0 \\ 0, & t \leq 0 \end{cases}$$

(b)

$$y(t) = \begin{cases} \frac{1}{t}, & t > 1 \\ 0, & t \leq 1 \end{cases}$$

(c)

$$y(t) = \begin{cases} e^{-t}(1 - e^{-t}), & t > 0 \\ 0, & t \leq 0 \end{cases}.$$

Solution

1.5

Consider the linear system

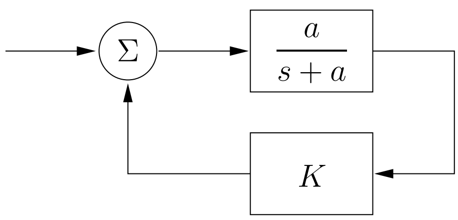
$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}.$$

Compute the system gain $\|G\|$ for all values of $\omega_0 > 0$ and $\zeta > 0$.

Solution

1.6

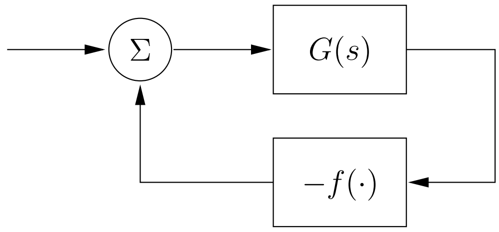
Analyze the stability of the following system, first by using the small gain theorem and then by computing the poles of the closed-loop system. Explain possible differences.



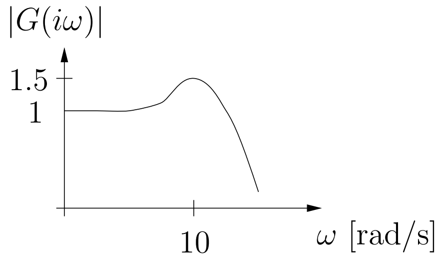
Solution

1.7

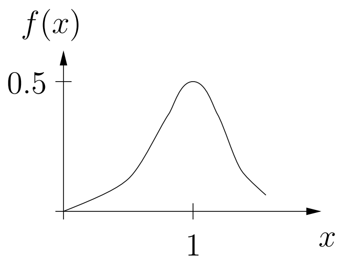
Consider the feedback control system



where $G(s)$ is a linear system with the magnitude plot



and $f(\cdot)$ s an amplifier with the following input-output relationship



Is the closed-loop system stable?

Solution

1.8

Consider a DC motor given on state-space form

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_2 + au \\ y &= x_1\end{aligned}$$

The inverse time constant a can vary as

$$a = 1 + \rho, \qquad |\rho| < \delta.$$

The system is controlled using a proportional controller $u(t) = -Ky(t)$. Suppose that a is constant. Give a sufficient condition on K such that the closed-loop system is stable for all a.

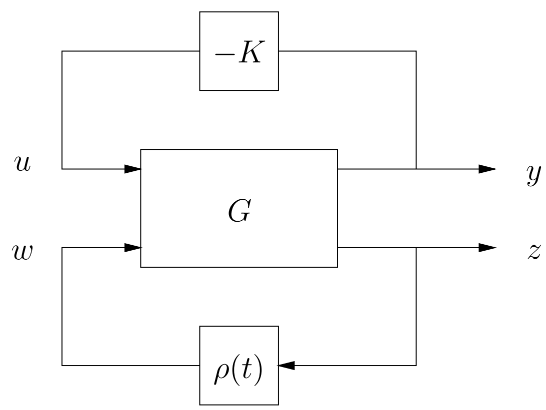
Solution

1.9

Once again consider the DC motor in exercise 1.8, but now assume that the parameter a can vary arbitrarily fast with time

$$a = a(t) = 1 + \rho(t), \qquad |\rho(t)| < \delta, \ \forall t$$

(a) Introduce a new, artificial input signal w and a new artificial output signal z such that the system can be described by the feedback connection below



(b) Consider the time-varying, static system from $z(t)$ to $w(t)$:

$$w(t) = \rho(t) \cdot z(t), \qquad |\rho(t)| < \delta,\ \forall t.$$

Show that the gain of this system (according to Definition 1.1 in the textbook) is at most δ.

(c) Give a sufficient condition on K, for instance an inequality that implicitly characterizes K, for the closed-loop system to be stable no matter how $a(t)$ varies with time.

Solution

2 Representation of Linear Systems

2.1

A simplified model of an alternating-current generator can be described as follows. The input signals to the system are the magnetizing current I_m, which is fed into the armature winding, and the driving torque M which is applied to the rotor axis. The rotation speed of the generator is ω, and the change in rotation speed is given by

$$J\dot{\omega} = M - M_e$$

where

$$M_e = K_e \cdot \omega \cdot I_f$$

is the electrical torque due to the emf. I_f is the current in the stator winding, given by the relationship

$$e = R \cdot I_f$$

where the voltage e is generated in the stator winding according to

$$e = C_e \cdot I_m \cdot \omega$$

and R is the load resistance applied to the stator winding. Consider e and ω as output signals, M, I_m and R as input signals. Set $K_e = C_e = J = 1$ and find a state-space representation for this sytem.

Linearize around the stationary point

$$\omega_0 = R_0 = I_{m0} = M_0 = 1$$

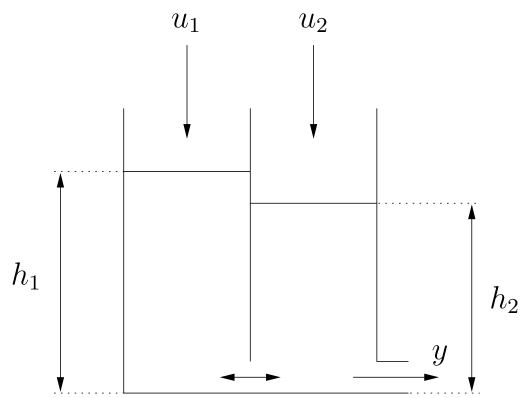
and derive the transfer function matrix from

$$u = \begin{bmatrix} \Delta M \\ \Delta I_m \\ \Delta R \end{bmatrix} \text{ to } y = \begin{bmatrix} \Delta\omega \\ \Delta e \end{bmatrix}$$

Solution

2.2

Consider the system consisting of two coupled tanks described in the figure below.



The flow of water into the left and right halves of the tank are denoted u_1 and u_2 respectively. These flows are the input signals. The water levels in the two halves are denoted h_1 and h_2 respectively. The flow y out from the tank is assumed to be proportional to the water level in the right half of the tank

$$y(t) = \alpha h_2(t)$$

The flow between the two halves is proportional to the difference between the levels

$$f(t) = \beta(h_1(t) - h_2(t))$$

where a flow from left to right is considered positive. Let h_i, u_i and y be deviations from nominal values. Thus, they can have negative values. Assume that the area of the halves are $A_1 = A_2 = 1$.

(a) Derive the transfer function from u_1, u_2 to y.

(b) Compute the maximum and minimum singular values of $G(0)$ and give an intuitive explanation to the corresponding input signals.

Solution

2.3

Find a state-space realization of the system

$$G(s) = \begin{bmatrix} \frac{1}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s^2+s+1)} \end{bmatrix}$$

Solution

2.4

Find a state-space realization of the system

$$y(t) = \frac{p}{p^2+4p+4} u_1(t) + \frac{p-1}{p^2+5p+6} u_2(t)$$

Solution

2.5

A system is described by the differential equation

$$\ddot{y} + a_1\dot{y} + a_2 y = b_{11}\dot{u}_1 + b_{12}u_1 + b_{21}\dot{u}_2 + b_{22}u_2.$$

Find a state-space realization.

Solution

2.6

Consider the system

$$\begin{cases} \dot{y}_1 + y_2 & = & \dot{u} + 2u \\ \dot{y}_2 + y_2 + y_1 & = & u \end{cases}$$

Find a state-space realization.

Solution

3 Properties of Linear Systems

3.1

Consider the transfer function matrix

$$G(s) = \begin{pmatrix} \frac{1}{s+2} & -\frac{1}{s+2} & \frac{1}{s+2} \\ \frac{1}{s+2} & \frac{s+1}{s+2} & \frac{1}{s+2} \end{pmatrix}$$

Derive the pole and the zero polynomials of the system? What is the dimension of a minimal state-space realization?

Solution

3.2

Find the poles and zeros of

$$G(s) = \frac{1}{(s+1)(s+3)} \begin{pmatrix} 1 & 0 \\ -1 & 2(s+1)^2 \end{pmatrix}$$

Solution

3.3

Find the poles of

$$G(s) = \frac{1}{(s+1)^2} \begin{pmatrix} 1-s & \frac{1}{3}-s \\ 2-s & 1-s \end{pmatrix}.$$

What is the dimension of a minimal realization?

Solution

3.4

(a) Consider the system

$$G(s) = \begin{pmatrix} \frac{s+5}{s^2+3s+2} & \frac{1}{s+2} \\ \frac{1}{s+4} & \frac{1}{s+2} \end{pmatrix}$$

What is the dimension of a minimal state-space realization?

(b) Consider the system

$$G(s) = \begin{pmatrix} \frac{s+5}{s^2+3s+2} & \frac{1}{s+2} \\ \frac{1}{s+4} & \frac{1}{s+4} \end{pmatrix}$$

What is the dimension of a minimal state-space realization?

Solution

3.5

A system has the following input-output relation

$$\begin{cases} \dot{y}_1 + y_1 - \dot{y}_2 &= u_1 - u_2 \\ \dot{y}_2 + \dot{y}_1 + y_2 &= u_1 + u_2 \end{cases} .$$

Find a matrix fraction description , $y(t) = A(p)^{-1}B(p)u(t)$, and compute the poles and zeros of the system.

Solution

3.6

Consider the MIMO system:

$$\dot{x}(t) = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x(t)$$

Find a minimal relization of the system, i.e. a realization that is controllable and observable.

Solution

3.7

Consider the multivariable system

$$Y(s) = G(s)U(s)$$

where

$$G(s) = \begin{pmatrix} \frac{1}{s+1} & \frac{3}{s+2} \\ \frac{2}{s+3} & \frac{1}{s+4} \end{pmatrix}$$

(a) Determine the maximum and minimum singular value of the frequency reponse at the frequency $\omega = 2$ rad/s.

(b) Determine also the input vectors, in terms of their Fourier transforms, corresponding to the largest and smallest gain of the system at $\omega = 2$.

(c) Generate, in Matlab, an input vector that corresponds to the largest gain of the system and simulate the system using this input.

Hint: Use sinusoidal input signals and use the following properties of a sinusoidal signal considered over a finite time interval.

- The Fourier transform is proportional to the amplitude of the sinusoidal signal, i.e. for $u_1(t) = A\sin\omega t$ the Fourier transform $U_1(i\omega)$ is proportional to A.
- Time delay of a signal corresponds to a change of the argument of the Fourier transform, i.e. if $u_1(t) = A\sin\omega t$ has the transform $U_1(i\omega)$ the signal $A\sin(\omega t + \phi)$ has the transform $U_1(i\omega)e^{i\phi}$.

(d) Verify that the obtained output signals correspond to largest gain of the system.

Solution

3.8

Find the poles and zeros of

$$G(s) = \begin{pmatrix} \frac{1}{s+1} & 0 & \frac{s-1}{(s+1)(s+2)} \\ \frac{-1}{s-1} & \frac{1}{s+2} & \frac{1}{s+2} \end{pmatrix}.$$

Solution

5 Disturbance Models

5.1

A continuous-time stochastic process $u(t)$ has the power spectrum $\Phi_u(\omega)$. For the power spectra below, find linear filters such that the processes can be represented as white noise fed through those filters.

(a) $$\Phi_u(\omega) = \frac{a^2}{\omega^2 + a^2}$$

(b) $$\Phi_u(\omega) = \frac{a^2 b^2}{(\omega^2 + a^2)(\omega^2 + b^2)}$$

Solution

5.2

A position sensor is mounted on a machine that vibrates with a frequency around 5 Hz, and this causes that a disturbance $n(t)$ affects the position measurement. In order to include the properties of the measurement disturbance in the control design one formulates a a model that describes the properties of the disturbance as filtered white noise V. The following models are suggested

$$(i) \quad N(s) = \frac{1}{s + 0.001} V(s)$$

$$(ii) \quad N(s) = \frac{900}{s^2 + 6s + 900} V(s)$$

$$(iii) \quad N(s) = \frac{25}{s^2 + s + 25} V(s)$$

Which disturbance model is the best choice?

Solution

5.3

Consider a missile propelled by the thrust u. The missile's position is z. A simplified model for the air drag is

$$f = k_1 \cdot \dot{z} + v$$

where v are, more or less, random wind gusts.

(a) Derive a state-space representation and an input-output representation for how the controlled output z depends on u and v.

(b) The system disturbance v has the spectral density

$$\Phi_v(\omega) = k_0 \cdot \frac{1}{\omega^2 + a^2}$$

Modify the state-space representation in (a) to make it possible to express the system disturbance using white noise. What is the corresponding transfer function?

Solution

5.4

Assume, in exercise 5.3, that the position z is measured with an error

$$y(t) = z(t) + n(t)$$

Derive a state-space model for the missile if

(a)

$$\Phi_n(\omega) = 0.1$$

(b)

$$\Phi_n(\omega) = 0.1 \frac{\omega^2}{\omega^2 + b^2}.$$

(c)

$$\Phi_n(\omega) = 0.1\frac{1}{\omega^2 + b^2}.$$

Solution

5.5

A system has the state-space representation

$$\begin{aligned}\dot{x} &= Ax + Bu + Nw \\ y &= Cx + n\end{aligned}$$

We assume that the system disturbance w changes stepwise and that the measurement noise is periodical with a frequency of about 2 Hz.

Modify the state-space representation to make it possible to model the disturbances.

Solution

5.6

In airplanes it is common to measure acceleration as well as speed. The acceleration is measured using accelerometers and the speed is calculated from measurements of air data, such as dynamical pressure et cetera. Thus, the measurements are independent, but of course they are related to each other.

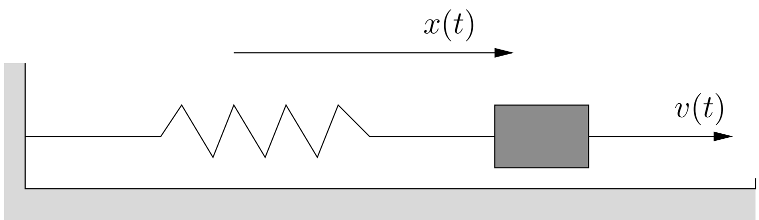
(a) Derive a state-space model for the speed and acceleration. Let the measured speed and acceleration be output signals and assume that the derivative of the acceleration is white noise. Furthermore, assume that the measurement errors in speed and acceleration are white noises, independent of each other.

(b) Discuss how we can get better estimates of the speed and acceleration using Kalman filtering.

Solution

5.7

The depicted dynamical system is described by the differential equation



$$\ddot{x}(t) + x(t) = v(t)$$

The external force $v(t)$ is white noise with

$$E\ v(t) = 0$$
$$E\ v(t)v(s) = \delta(t-s)$$

We want to estimate the position $x(t)$ and speed $\dot{x}(t)$ at every time instant. We have sensors for both speed and position but for economical reasons we only want to use one sensor. We can choose between

Alternative I: The measured signal is

$$y_1(t) = x(t) + e_1(t)$$

Alternative II: The measured signal is

$$y_2(t) = \dot{x}(t) + e_2(t)$$

The measurement errors are $e_1(t)$ and $e_2(t)$. For simplicity we assume that they are both white noises with

$$Ee_1(t) = Ee_2(t) = E[e_1(t)e_2(s)] = 0$$

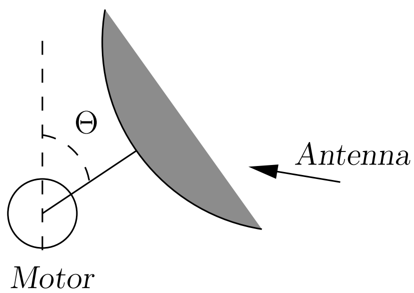
$$Ee_1(t)e_1(s) = Ee_2(t)e_2(s) = \delta(t-s)$$

For each alternative derive the linear filter that, in steady state, yields the best estimate of $x(t)$ and $\dot{x}(t)$, in the sense of smallest variance of the estimation error, from measurements up to and including time t. State, with an explaination, which alternative you think is the best.

Solution

5.8

Consider the depicted radar antenna.



From noisy measurments of the position of the antenna Θ_m we want to estimate the true position Θ. To be able to do this we need a model of the system. To this end, describe the dynamics of the antenna with

$$J\ddot{\Theta}(t) + B\dot{\Theta}(t) = \tau(t) + \tau_d(t),$$

where J is the moment of inertia for the moving parts of the antenna, B is the coefficient of viscous friction, $\tau(t)$ is the torque produced by the motor, and $\tau_d(t)$ is the torque caused by the wind. Assume that $\tau_d(t)$ can be modeled as white noise. Furthermore, assume that the torque $\tau(t)$ is proportional to the motor voltage , $\mu(t)$, i.e.

$$\tau(t) = k\mu(t)$$

Finally, let us for simplicity, assume that the measurement error can be modeled as additative white noise $e_m(t)$. Hence, the output signal is

$$\Theta_m(t) = \Theta(t) + e_m(t).$$

Discuss how $\Theta(t)$ can be estimated from $\Theta_m(t)$ using a Kalman filter.

Technical data:

$$B/J = 4.6\ s^{-1}$$

$$k/J = 0.787\ rad/Vs^2$$

$$J = 10\ kg\ m^2$$

$$E\ \tau_d(t)\tau_d(s) = v_d\delta(t-s) = 10\ N^2m^2 \cdot \delta(t-s)$$

$$E\ e_m(t)e_m(s) = v_m\delta(t-s) = 10^{-7}\ rad^2 \cdot \delta(t-s)$$

Solution

5.9

Consider an electric motor with transfer operator

$$G(p) = \frac{1}{p(p+1)}$$

from input voltage to actual angular displacement. The motor operates in two disturbance modes:

(i)

$$y(t) = G(p)(u(t) + w(t))$$

(ii)

$$y(t) = G(p)u(t) + w(t)$$

In both cases we have $w(t) = \frac{1}{p}v(t)$ where $v(t)$ is a unit disturbance, for example an impulse.

(a) Realize both cases on state-space form. For case (ii) it is assumed that the states caused by the disturbance are separate from the ones describing the motor dynamics.

(b) For both cases, give examples of physical phenomena that can be modeled with the disturbance $w(t)$.

(c) Study the two state-space realizations. Are all states controllable? Can states corresponding to $w(t)$ be made unobservable? Can the influence of $w(t)$ on $y(t)$ be eliminated?

Solution

5.10

Consider the movement of a swing due to the wind. The swing is descibed by the transfer operator

$$y(t) = \frac{1}{p^2 + p + 1} u(t)$$

where the output signal $y(t)$ is the angular displacement and the input signal $u(t)$ is the torque about the point of suspension. The influence of the wind can be modeled as

$$u(t) = Kv(t)$$

where $v(t)$ is a Gaussian distributed disturbance with the spectrum

$$\Phi_v(\omega) = \frac{2\alpha}{\alpha^2 + \omega^2}, \qquad \alpha > 0.$$

K quantifies the strength of the wind and α quantifies the gustiness of the wind.

(a) Does α increase or decrease when the gustiness increases, i.e. when the wind changes direction more frequently?

(b) Derive and interprete conditions on α and K such that the swing has an angular displacement of more than 1.15 at least a quarter of the time. This is equivalent to the output having a variance greater than

1.

Hint:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|b_2(i\omega)^2+b_1 i\omega+b_0|^2}{|(i\omega)^3+a_2(i\omega)^2+a_1 i\omega+a_0|^2}\,d\omega$$
$$=\frac{b_2^2a_0a_1+(b_1^2-2b_0b_2)a_0+b_0^2a_2}{2a_0(-a_0+a_1a_2)}$$

Solution

6 The Closed-Loop System

6.1

For a given system G and a given controller F we have defined four transfer functions as

$$\begin{aligned} G_{w_u u} &= (I + FG)^{-1}, \qquad & G_{wu} &= -(I + FG)^{-1} F \\ G_{w_u y} &= (I + GF)^{-1} G, \qquad & G_{wy} &= (I + GF)^{-1} \end{aligned}$$

All four transfer functions have to be stable for the closed-loop system to be internally stable.

Show that

$$\begin{pmatrix} G_{w_u u} & G_{wu} \\ G_{w_u y} & G_{wy} \end{pmatrix} = \begin{pmatrix} I & F \\ -G & I \end{pmatrix}^{-1}$$

Solution

6.2

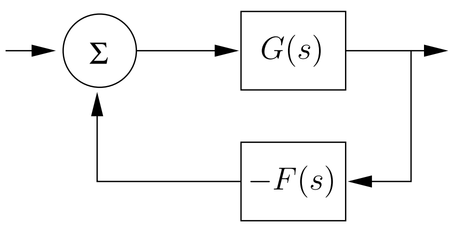
The system

$$G(s) = \frac{s-1}{s+1}$$

and the controller

$$F(s) = \frac{s+2}{s-1}$$

are are used in the feedback connection depicted below.



Compute G_c, T and S. Are they stable? Is the closed-loop system internally stable?

Solution

7 Limitations in Control Design

7.1

Given the system

$$G(s) = \frac{s-3}{s+1}.$$

we want the complementary sensitivity function to be

$$T(s) = \frac{5}{s+5}.$$

(a) Compute a controller $F_r = F_y = F$ which results in this T. Will this controller really work?

(b) Suggest an alternative T, still having the bandwidth 5 rad/s, but resulting in an internally stable system with $F_r = F_y = F$.

(c) A rule of thumb for control of non-minimum phase systems states that the bandwidth of the closed-loop system cannot realistically be greater than half the value of the non-minimum phase zero. In this case 1.5 rad/s. Have we cirumvented this rule of thumb in the above design or does the closed-loop system have any disadvantages?

Solution

7.2

A continuous-time system has a zero at $s = 3$ and a time-delay of 1.0 second. What is the upper limit of the realistic bandwith/crossover frequency if the magnitude curve of the open-loop system decreases monotonically?

Solution

7.3

Give an example of a system for which there exists no controller having all three properties: a stable closed-loop system, small magnitude of the sensitivity function at low frequencies and small amplification of measurement errors at high frequencies.

Solution

7.4

A multivariable system is supposed to attenuate system disturbances (w) at least a factor 10 for frequencies under 0.1 rad/s. Furthermore, measurement disturbances (n) should be attenuated at least a factor 10 for frequencies above 2 rad/s. Constant system disturbances should be attenuated at least a factor 100 in steady state.

(a) Formulate conditions on the singular values of S and T which will guarantee that the requirements are fulfilled.

(b) Translate the specifications into requirements on the loop gain GF_y.

(c) Formulate the requirements using $\|\cdot\|_\infty$ and frequency weights W_S och W_T.

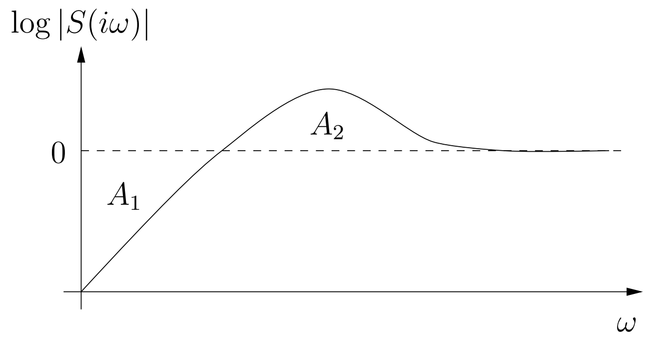
(d) Which crossover frequency and phase margin would we expect, having the weights i (b), had the system been a SISO system? What lower bound on $\|T\|_\infty$ does this result in?

(e) Is this lower bound on $\|T\|_\infty$ consistent with the requirements in (c)?

Solution

7.5

A control system has the sensitivity function S, depicted below



What can be stated about the open-loop system if the surface A_2 is larger than the surface A_1?

Solution

7.6

For a certain feedback system we demand that:

(i) output disturbances, with frequencies under 2 rad/s, should be attenuated at least a factor 1000.

(ii) the system should remain stable despite a model uncertainty

$$|\Delta G| \leq 100|G|$$

for frequencies above 20 rad/s. G is the frequency response of the nominal system and ΔG is the absolute error in the frequency response.

Can this be accomplished using a linear, time-invariant controller?

Solution

7.7

We have the following specifications on a SISO system

$$\begin{aligned}|S(i\omega)| &\le 10^{-3}, \qquad \omega \le 1\\ |T(i\omega)| &\le 10^{-3}, \qquad \omega \ge 100\end{aligned}$$

(a) State two non-constant frequency weights W_S and W_T which would guarantee that the specifications are met.

(b) Trying to find a controller fulfilling the design criteria, for example using the methods presented in Chapter 10 in the textbook, we fail. Should this have been anticipated from the very beginning?

Solution

8 Controller Structure and Control Design

8.1

Let

$$G(s) = \begin{pmatrix} \frac{1}{s+2} & \frac{10}{s+1} \\ \frac{1}{s+5} & \frac{5}{s+3} \end{pmatrix}.$$

(a) Compute RGA($G(0)$).

(b) Which input-output pairing should be avoided?

Solution

8.2

Given the multivariable system

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{0.1s+1} \begin{pmatrix} \frac{0.6}{s+1} & -0.4 \\ 0.3 & 0.6 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Assume that we want the controller to be diagonal and that we use the *relative gain array* (RGA) to decide what input should control what output. Furthermore, assume that we want a crossover frequency of $\omega_c = 10$ rad/s. Decide how the signals should be paired.

Solution

8.3

Study the multivariable system

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{10s+1} & \frac{-2}{2s+1} \\ \frac{1}{10s+1} & \frac{s-1}{2s+1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

(a) Decide, using RGA analysis, which input signal should control which output signal.

(b) Assume that we want to use decentralized control, i.e. we want a controller on the form

$$F = W_1 F^{\text{diag}}(s) W_2, \text{ where } F^{\text{diag}}(s) = \begin{pmatrix} F_{11}(s) & 0 \\ 0 & F_{22}(s) \end{pmatrix}.$$

Furthermore, assume that we do not want the steady-state error in one channel to affect the steady-state error in the other channel. Give the structure of a controller $F(s)$, expressed in $F^{\text{diag}}(s)$, that will accomplish this.

Solution

8.4

Design a controller, using the IMC method, for a stable first order process

$$G(s) = \frac{K}{\tau s + 1}, \qquad \tau > 0.$$

What type of controller do we get? Compute the sensitivity function and the complementary sensitivity function and sketch the Bode plot of the sensitivity function. What does Bode's integral theorem state for this case?

Solution

8.5

Design a controller, using the IMC method, for the system

$$G(s) = \frac{6 - 3s}{s^2 + 5s + 6}.$$

What type of controller do we get?

Solution

8.6

Consider the DC motor

$$y = \frac{1}{p(p+1)} u$$

Compute an IMC based controller for this system. Write the controller on the form $u = -F_y(p)y$, and sketch the Bode plot for $F_y(p)$. Approximately what type of controller do we get when we want a high bandwidth for the closed-loop system?

Solution

8.7

Given the multivariable system

$$G(s) = \frac{1}{s/20+1} \begin{pmatrix} \frac{9}{s+1} & 2 \\ 6 & 4 \end{pmatrix}.$$

(a) What are the poles and zeros of $G(s)$?

(b) Compute an IMC based controller for the system.

Solution

8.8

Consider the system

$$G(s) = \begin{pmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix}$$

(Example 1.1 in the textbook)

Show how an IMC based controller can be computed for this system. Give an explicit expression for the corresponding sensitivity function.

Solution

8.9

Consider the multivariable system

$$Y(s) = G(s)U(s)$$

where

$$G(s) = \begin{pmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{\alpha}{s+1} & \frac{1}{s+1} \end{pmatrix}$$

and $\alpha > 0$.

(a) Determine the zero of the multivariable system. How does the zero depend on the value of α?

(b) Assume that one would like to achive complete decouple of the system $G(s)$ such that

$$G(s)F(s) = \begin{pmatrix} \frac{1}{(s+1)^2} & 0 \\ 0 & \frac{1}{(s+1)^2} \end{pmatrix}$$

Are there any cases when this is not a good idea? Motivate!

(c) Assume that one instead chooses to use a static decoupling such that $G(s)F(s)$ is decoupled for $\omega = 0$. Are there any values of α for which this is not a good idea? Motivate!

Solution

8.10

Consider the multivariable system

$$Y(s) = G(s)U(s)$$

where

$$G(s) = \begin{pmatrix} \dfrac{1}{s+2} & \dfrac{2}{s+4} \\ \dfrac{1}{s+1} & \dfrac{1}{s+2} \end{pmatrix}$$

(a) Determine the RGA at $\omega = 0$.

(b) Assume that the system is going to be controlled by the diagonal regulator

$$U(s) = F(s)(R(s) - Y(s))$$

where

$$F(s) = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$$

Use the result from a) the judge how successful this will be. Determine also the poles of the closed loop system for the case $K = 5$.

(c) How can the controller (or system model) be modified such that a diagonal (one input controls one output) $F(s)$ can be used? Verify that the closed loop system is stable for $K = 5$ for the modified setup.

Solution

9 Minimization of Quadratic Criteria: LQG

9.1

Consider the system

$$G(s) = \frac{1}{s-1}$$

represented on state-space form with noise as

$$\begin{aligned}
\dot{x}(t) &= x(t) + u(t) + v_1(t) \\
z(t) &= x(t) \\
y(t) &= x(t) + v_2(t)
\end{aligned}$$

The noises $v_i(t)$ are white with intensities R_i. We use the criterion

$$V = \int Q_1 x^2(t) + Q_2 u^2(t)\, dt,$$

and want to find the LQG controller.

(a) Show that the controller is a function of $\alpha = Q_1/Q_2$ and $\beta = R_1/R_2$ only.

(b) Compute the poles of the closed-loop system as a function of α and β.

Solution

9.2

Consider the system

$$\begin{aligned}
z &= \frac{1}{p+1}u + \frac{1}{p+1}v \\
y &= z + e
\end{aligned}$$

where v and e are unit disturbances with spectra

$$\Phi_v(\omega) \equiv r_1 \quad \text{respektive} \quad \Phi_e(\omega) \equiv 1.$$

We minimize the criterion

$$V = \int q_1 z^2(t) + u^2(t)\, dt$$

(a) Compute the loop gain of the feedback connection.

(b) How do r_1 and q_1 influence the loop gain?

(c) Sketch the magnitude of the frequency response. What happens when $r_1 \to \infty$ and when $q_1 \to \infty$ respectively?

Solution

9.3

Consider the double integrator

$$\ddot{z}(t) = u(t).$$

We want to find a controller such that the criterion

$$\int_0^{\infty} (z^2(t) + \eta \cdot u^2(t))\, dt$$

is minimized for some $\eta > 0$. We assume that $z(t)$ and $\dot{z}(t)$ are both known (and need not to be estimated).

Where are the poles of the optimal closed-loop system located? How is the control signal affected when η is decreased?

Solution

9.4

Consider the antenna in Exercise 5.8. We want to control it and a suitable measure on the performance of closed-loop system is given by the criterion

$$J = E\left\{\Theta^2(t) + \rho\mu^2(t)\right\}$$

where ρ is a constant we can choose. Derive an optimal control signal and discuss how it is to be combined with the Kalman filter.

Solution

9.5

Consider control of the DC motor

$$G(s) = \frac{1}{s(s+1)}$$

We want to use the motor together with a system that has a resonance peak at approximately 0.5 rad/s. Other than that, we do not know much about the system. Describe how we can compute an LQG controller with good robustness qualities, i.e. small complementary sensitivity gain, at this frequency.

Solution

9.6

A system has static gain G_0. It is influenced by system disturbances, with all energy concentrated at zero frequency, i.e.

$$\Phi_\nu(\omega) = \delta(\omega)$$

The reference signal is zero, as is the measurement noise. We choose a controller that minimizes

$$E\left\{y^2(t) + \alpha u^2(t)\right\}$$

What is the value of the sensitivity function at zero frequency?

Solution

9.7

Consider the system

$$\begin{aligned} z &= \frac{1}{p+1}u + \frac{1}{p+1}\nu \\ y &= z + e \end{aligned}$$

where ν is noise of very low frequency,

$$\nu = \frac{1}{p+\varepsilon}v,$$

v and e are noises with $\Phi_v(\omega) \equiv \Phi_e(\omega) \equiv 1$.

(a) Find a controller that minimizes

$$E\left\{z^2 + u^2\right\}$$

when $\varepsilon \to 0$.

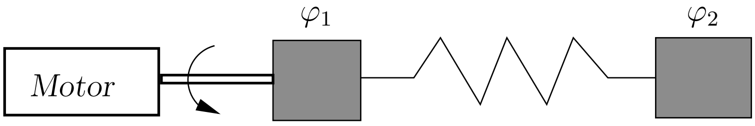
What is the static gain of the sensitivity function?

(b) Use output-LTR (LTR(y)) to compute L. What is the static gain of the sensitivity function?

Solution

9.8

Consider a motor driving two rotating masses connected by a flexible shaft:



The angular displacements of the masses are φ_1 and φ_2 respectively and ω_1 and ω_2 are the angular velocities. The moments of inertia are 10 for both masses. The spring rate of the shaft is k and the damping factor is 0.1. The input is the voltage applied to the motor. With the states $x_1 = \varphi_1 - \varphi_2$, $x_2 = \omega_1$ and $x_3 = \omega_2$ we get the state-space representation

$$\dot{x} = \begin{pmatrix} 0 & 1 & -1 \\ -\frac{1}{2}\omega_0^2 & -0.01 & 0.01 \\ \frac{1}{2}\omega_0^2 & 0.01 & -0.01 \end{pmatrix} x + \begin{pmatrix} 0 \\ \omega_0 \\ 0 \end{pmatrix} u$$

$$z = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x$$

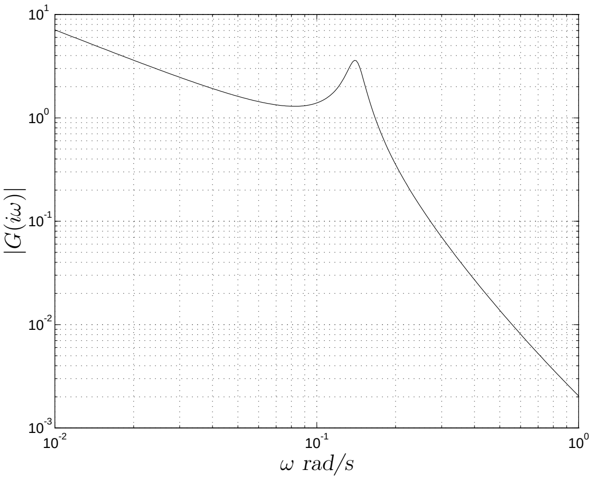
where

$$\omega_0^2 = \frac{k}{50}$$

The Bode plot, when $k = 1$, is shown in the figure below. There is a resonance peak at the frequency ω_0. The spring rate is not exactly known, but has a value close to 1. We want to design a controller that yields a stable closed-loop system despite variations in k.

How can the above model be extended with a model for the noise to assure robustness for an uncertain value of k when we use LQG controller design? Give an actual example of such an extended system.

Solution



9.9

Consider the system

$$\dot{x}(t) = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 3 \\ 2 \end{pmatrix} u(t)$$

Show that

$$u(t) = -\begin{pmatrix} 2 & -3 \end{pmatrix} x(t)$$

cannot be an optimal state feedback for any quadratic criterion on the form

$$\min \int (x^T(t)Q_1x(t) + Q_2u^2(t))\, dt$$

where Q_1 is a positive definite matrix.

Solution

9.10

Consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} x + \begin{pmatrix} -4 \\ 8 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 1 \end{pmatrix} x \end{aligned}$$

We want to minimize the criterion

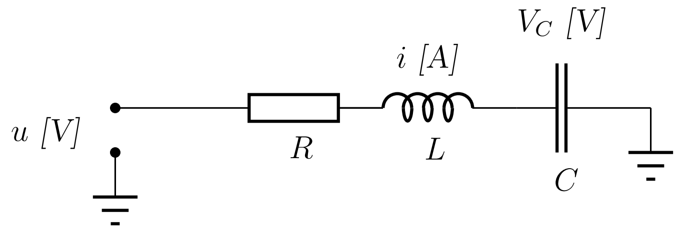
$$V(T) = \int_0^T x^T(t)x(t) + u^2(t)dt$$

Is it possible to find a state feedback $u = -Lx$ such that $V(T) < \infty$ when $T \to \infty$?

Solution

9.11

The figure below shows a simple electrical circuit.



Introduce the state variables $x_1 = V_C$ and $x_2 = i$. With the component values

$$R = 5\ \Omega, \qquad L = 0.1\ H, \qquad C = 1000\ \mu F$$

we get the state-space representation

$$\dot{x}(t) = \begin{pmatrix} 0 & 1000 \\ -10 & -50 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 10 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t)$$

Compute a state feedback that minimizes

$$J = \int_0^\infty \left(x_2^2(t) + 0.01u^2(t)\right)\, dt$$

This criterion aims at limiting the power loss without getting too large signals.

Solution

9.12

A system has the state-space representation

$$\dot{x}(t) = \begin{pmatrix} -2 & -1 & 1 \\ 1 & 0 & 0.5 \\ 0 & 0 & A \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v(t)$$

$$z(t) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x(t)$$

$$y(t) = z(t) + e(t)$$

where $e(t)$ and $v(t)$ are unit disturbances.

The controller, a feedback from reconstructed states, minimizes

$$E\left[z^2(t) + u^2(t)\right]$$

How does the value of A affect the sensitivity function?

Solution

9.13

A simplified model for how the elevator angle affects the movements of an airplane is given by

$$\dot{x} = \begin{bmatrix} -0.01 & 0.03 & -10 \\ 0 & -1 & 300 \\ 0 & 0 & -0.5 \end{bmatrix} x + \begin{bmatrix} 4 \\ -20 \\ -10 \end{bmatrix} u$$

where

$$x = \begin{bmatrix} \text{roll angle} \\ \text{yaw angle} \\ \text{pitch-angle velocity} \end{bmatrix}$$

In particular we are interested in the control of the pitch-angle velocity and choose the controlled variable to be

$$z = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x$$

All state variables are measured

$$y = x + e$$

We want to design a feedback from reconstructed states using LQG methodology. It is especially important that the sensitivity function has a small gain for frequencies around 1 rad/s. Show how to modify the model of the airplane to achieve such a sensitivity function.

Solution

9.14

Consider the system

$$\dot{x}(t) = \alpha x(t) + u(t) \qquad x(0) = x_0 \tag{1}$$

The system is controlled by the feedback

$$u(t) = -Lx(t) \tag{2}$$

where L is chosen such that

$$J = \int_0^\infty x^2(t) + \rho u^2(t))dt \tag{3}$$

is minimized.

(a) Determine L as function of ρ and α.

(b) If it is desired to keep $u(t)$ small, this can be achieved by choosing ρ large. What is the resulting L when $\rho \to \infty$? Consider, for example, the cases $\alpha = 1$ and $\alpha = -1$, respectively. Why is it not optimal to choose $L = 0$, i.e. $u(t) = 0$, in both cases?

Solution

9.15

An electrical motor has the transfer functions

$$Y(s) = \frac{1}{s(s+1)}U(s)$$

and it is controlled using state feedback

$$u(t) = -Lx(t) \qquad (r(t) = 0)$$

where $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$. The gain vector L is determined by minimizing the criterion

$$J = \int_0^\infty x^T(t)Q_1x(t) + Q_2u^2(t)dt$$

Figure 1 shows the simulation results when the system starts in the initial condition $x(0) = (1 \;\; 1)^T$ for some different choices of Q_1 and Q_2. Combine the figures with the choices of matrices.

(i)

$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad Q_2 = 0.1$$

(ii)

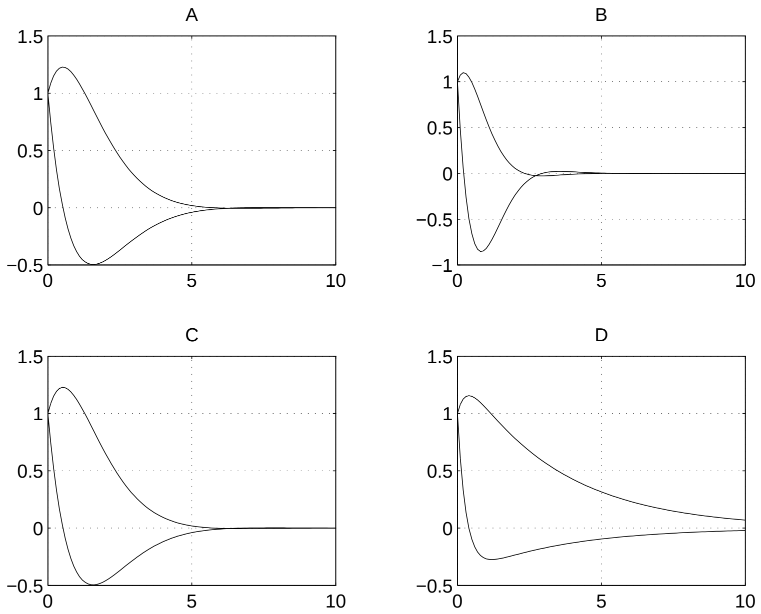
$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix} \qquad Q_2 = 1$$

(iii)

$$Q_1 = \begin{pmatrix} 0.1 & 0 \\ 0 & 0 \end{pmatrix} \qquad Q_2 = 0.1$$

(iv)

$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad Q_2 = 1$$

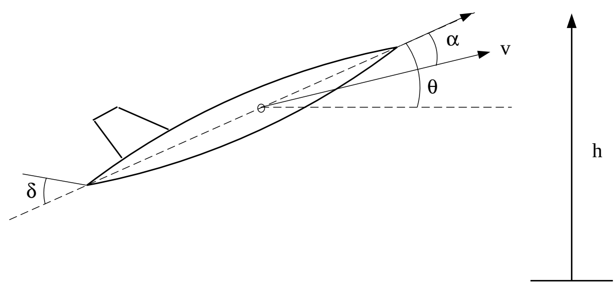


Figur 1:

Solution

9.16

Consider the simplified description of an aircraft in the figure below.



Using the state space variables

$$\begin{array}{ll} x_1(t) = \alpha(t) & \text{angle of attack (rad)} \\ x_2(t) = \dot{\theta}(t) & \text{pitch rate (rad/s)} \\ x_3(t) = \theta(t) & \text{pitch angle (rad)} \\ x_4(t) = h(t) & \text{height (deviation from an operating point)} \end{array}$$

the input signal

$$u(t) = \delta(t) \quad \text{control surface angle (rad)}$$

and the primary state to be controlled is the height $h(t)$

The dynamics is decribed by

$$\dot{x} = Ax + Bu$$

where

$$A = \begin{pmatrix} -0.17 & 1 & 0 & 0 \\ -0.56 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2.22 & 0 & 2.22 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0.011 \\ 0.56 \\ 0 \\ 0 \end{pmatrix}$$

(a) Is the system asymptotically stable?

(b) Assume that the system has the initial state

$$x_0 = (0 \;\; 0 \;\; 0.1 \;\; 1)^T$$

and that the system is controlled by the state feedback)

$$u = -Lx + r$$

(with reference r assumed to be 0) where the gain vector L is chosen such that the criterion

$$\int_0^\infty x^T(t) Q_1 x(t) + u^T(t) Q_2 u(t)\, dt$$

is minimized. Assume that the matrices are chosen as

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q_2 = 1$$

Determine the poles of the closed loop system. Simulate the closed loop system and study both the states and the control signal.

(c) Assume that Q_2 is varied. How does that affect the location of the closed loop poles and the properties of x and u?

(d) Assume now that the following conditions shall be fulfilled:

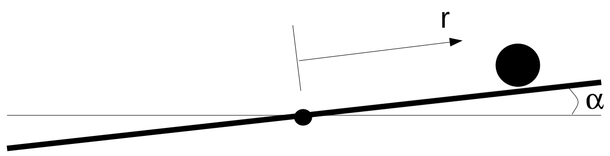
- $|x_1| < 0.2$ all the time.
- $|x_4| < 0.1$ after 25 seconds.
- $|u| < 0.5$ after one second.

Determine Q_1 and Q_2 such that these conditions are satisfied. What is the resulting location of the closed loop poles?

Solution

9.17

The figure below illustrates a system consisting of a ball on a plane. The variable r denotes the position of the ball relative to the center of the plane, and α represents the angle of the plane. The input signal is the torque that rotates the plane.



Figur 2: Ball on plane.

The system is represented by the state variables

$x_1(t)$	-	position, $r(t)$
$x_2(t)$	-	velocity, $\dot{r}(t)$
$x_3(t)$	-	plane angle, $\alpha(t)$
$x_4(t)$	-	plane angular velocity, $\dot{\alpha}(t)$

and torque is the input signal $u(t)$. The state space model is

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$$

(a) Assume that the system starts in the initial state

$$x(0) = (0.1 \;\; 0 \;\; -0.1 \;\; 0)^T$$

i.e. the ball is positioned to the right of the center, and the plane leans downwards on the right side. Assume that all state variables can be measured. Determine a state feedback such that the following requirements are fulfilled:

- $\mid x(t) \mid \rightarrow 0$ when $t \rightarrow \infty$.
- $\mid x_1(t) \mid \leq 0.2 \quad \forall\, t$.
- $\mid u(t) \mid \leq 2.5 \quad \forall\, t$

Determine also the absolute value of the poles of the closed loop system.

(b) Verify that all sensors that measure the states have to work in order to obtain a stable closed loop system.

Hint: The characteristic equation of the closed loop system is given by

$$\lambda^4 + l_4\lambda^3 + l_3\lambda^2 - 7l_2\lambda - 7l_1 = 0$$

Missing a sensor is equivalent to setting the corresponding feedback l_i to zero.

Solution

10 Loop Shaping

10.1

Consider the system

$$y = \frac{1}{p+1}u$$

We want to create a closed-loop system with S, T and G_{wu}, such that

$$\int \left|\frac{S(i\omega)}{i\omega}\right|^2 + |0.5\,T(i\omega)|^2 + |5\,G_{wu}(i\omega)|^2\,d\omega$$

is minimized. Compute the controller.

Solution

10.2

Consider the system

$$y = \frac{1}{p+1}u$$

We want to create a closed-loop system with S, T and G_{wu}, such that

$$|S(i\omega)| < \gamma\omega$$
$$|T(i\omega)| < 2\gamma$$
$$|G_{wu}(i\omega)| < 0.2\gamma$$

Write down the equations that determines the controller.

Solution

10.3

Consider the SISO system $G(s)$ with state-space realization

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

We want to use loop shaping with the weights

$$W_S = \frac{1}{s}, \quad W_T = 1, \quad W_u = 1$$

(a) State the equations that determine the optimal controller in $\mathcal{H}_2$ and $\mathcal{H}_\infty$ respectively.

(b) Explicitly write down the observer for the extended state vector and show that the optimal controller can be written as

$$u(t) = \frac{\alpha}{1 + L(pI - A)^{-1}B} \int_0^t y(\tau)\, d\tau$$

for some L, where $\alpha = 1$ for the $\mathcal{H}_2$ controller and $\alpha > 1$ for the $\mathcal{H}_\infty$ controller. State the equation determining L.

(c) Show that the controller will have a pole at the origin unless the system does itself has a pole at the origin.

Solution

10.4

Once again consider the system in Exercise 9.8.

(a) Suggest frequency weights W_S, W_T and W_u, for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ design, such that we get robustness against uncertain values of k.

(b) State the extended system from u and w to z on state-space form.

Solution

10.5

A DC-motor has transfer function

$$G(s) = \frac{1}{s(s+1)}$$

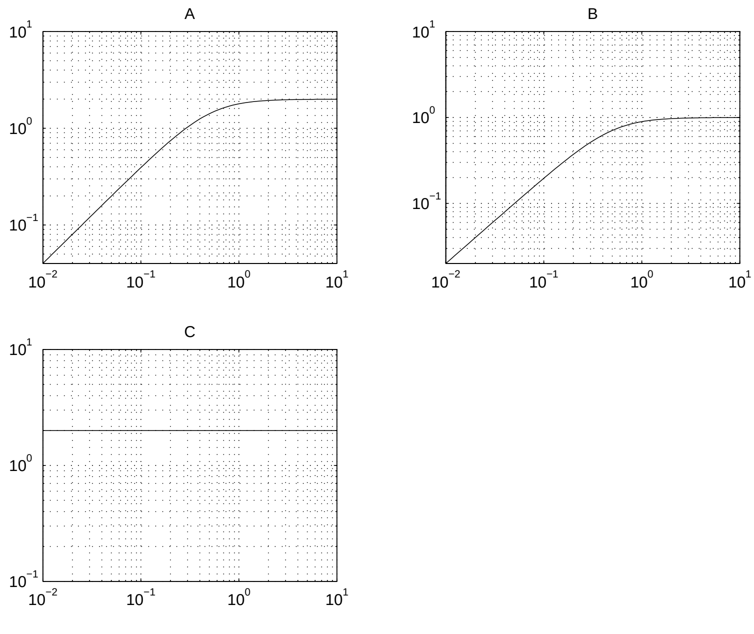
and it going to be controlled using proportional feedback

$$U(s) = K(R(s) - Y(s))$$

The properties of the closed loop system is specified via the requirement

$$\mid S(i\omega) \mid < \mid W_S^{-1}(i\omega) \mid \quad \forall\ \omega$$

The figure below shows three alternatives for the weight function $W_S^{-1}(i\omega)$. Which alternative is the best? Motivate the answer.



Figur 3: Suggestions for $\mid W_S^{-1}(i\omega) \mid$.

Solution

10.6

The system

$$Y(s) = \frac{1}{s+1}U(s)$$

is going to be controlled by the proportional feedback

$$U(s) = K(R(s) - Y(s))$$

(a) Derive $S(s), T(s)$ and $G_{ru}(s)$ respectively, i.e. the sensitivity function, the complementary sensitivity function and the transfer function from reference to input signal.

(b) The properties of the control system are specified using the weight function according to

$$\mid S(i\omega)W_S(i\omega) \mid < 1 \quad \forall \omega$$

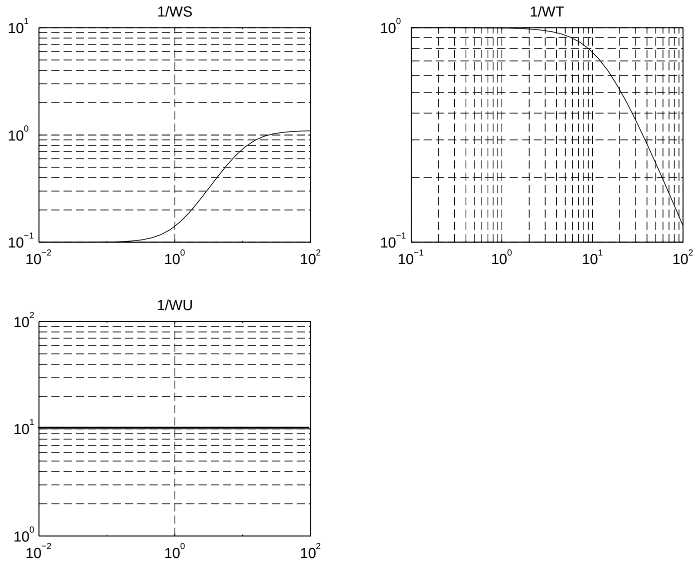
$$\mid T(i\omega)W_T(i\omega) \mid < 1 \quad \forall \omega$$

$$\mid G_{ru}(i\omega)W_u(i\omega) \mid < 1 \quad \forall \omega$$

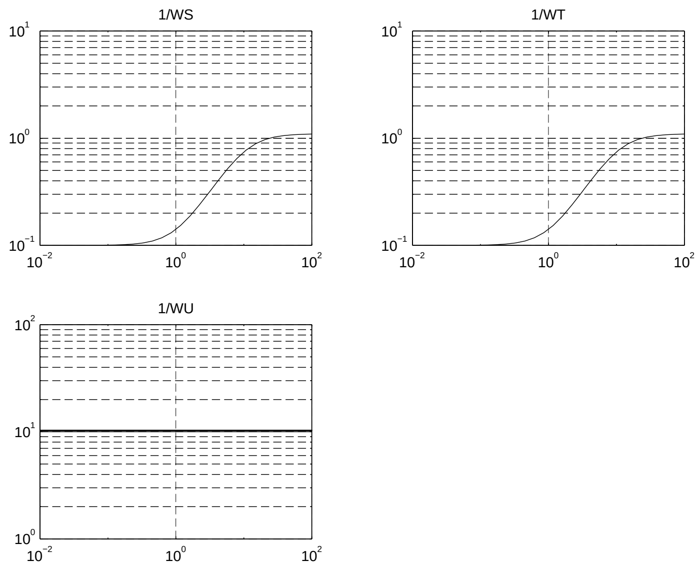
The figures below show three suggestions for weight functions W_S, W_T and W_U. Two of the alternatives are unrealistically or incorrectly specified. Which are the two incorrect alternatives? Motivate the answer.

(c) Consider the alternative in b) that is realistically specified. Is it possible to choose K such that all requirements are fulfilled?

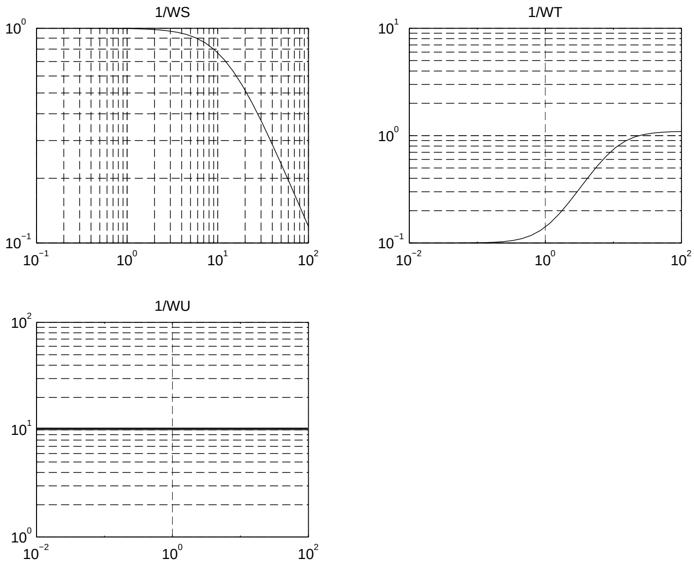
Solution



Figur 4: Alternative I



Figur 5: Alternative II



Figur 6: Alternative III

12 Stability of Nonlinear Systems

12.1

Given the nonlinear differential equation

$$\ddot{y} + 0.2(1 + \dot{y}^2)\dot{y} + y = 0$$

let the state variables be $x_1 = y$ and $x_2 = \dot{y}$. Try to show that the origin is a stable equilibrium by using the Lyapunov function candidate

$$V = \frac{1}{2}(x_1^2 + x_2^2).$$

Solution

12.2

Consider the system

$$\begin{aligned}\dot{x}_1 &= \sin x_1 + x_2^3 \\ \dot{x}_2 &= x_1 - x_2\end{aligned}$$

Is it possible to use the function

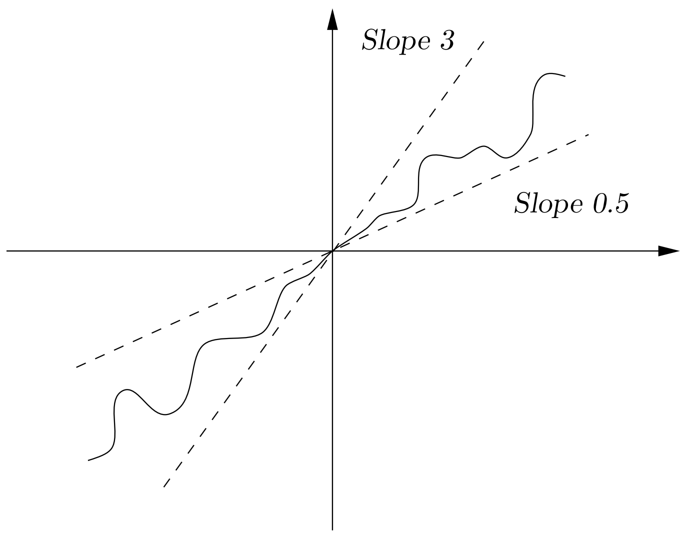
$$V(x_1, x_2) = -\frac{1}{2}x_1^2 + \frac{1}{4}x_2^4$$

to prove Lyapunov stability for the above system? Motivate your answer.

Solution

12.3

A nonlinear function lies in the sector

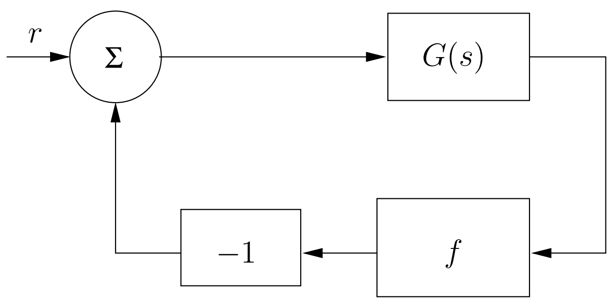


According to the circle criterion, what circle in the complex plane corresponds to this nonlinearity?

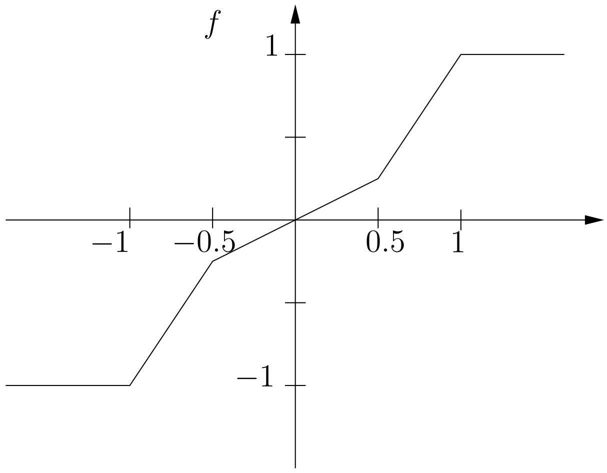
Solution

12.4

A nonlinear system is described by the following block diagram



where $G(s)$ is a linear system and the static nonlinearity f is given in the figure below (the saturations at -1 and 1 extends to $-\infty$ and ∞).

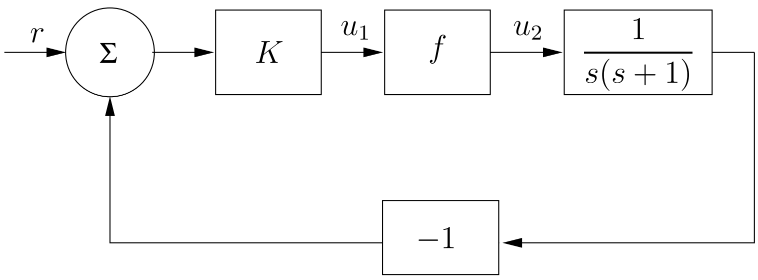


What assumptions on $G(s)$ must be fulfilled in order to prove that the feedback system is stable according to the circle criterion?

Solution

12.5

Consider the system below.

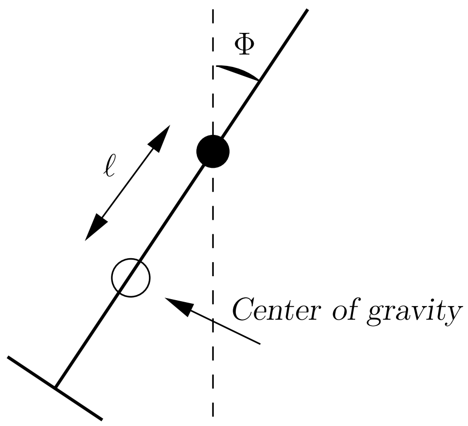


The nonlinearity f is such that u_2 has the same sign as u_1 but is otherwise not known. For what values of $K > 0$ is the feedback system stable according to the circle criterion?

Solution

12.6

Consider the swing depicted below.



The movement of the swing is described by the equation

$$J\frac{d^2\Phi}{dt^2} + mg\ell \sin\Phi = 0$$

where m is the mass and J is the moment of inertia. The swing can be controlled by alternating between bending and stretching the knees while standing on the swing. The control signal is the location of the center of gravity ℓ. We assume that J is constant.

Show that the control signal

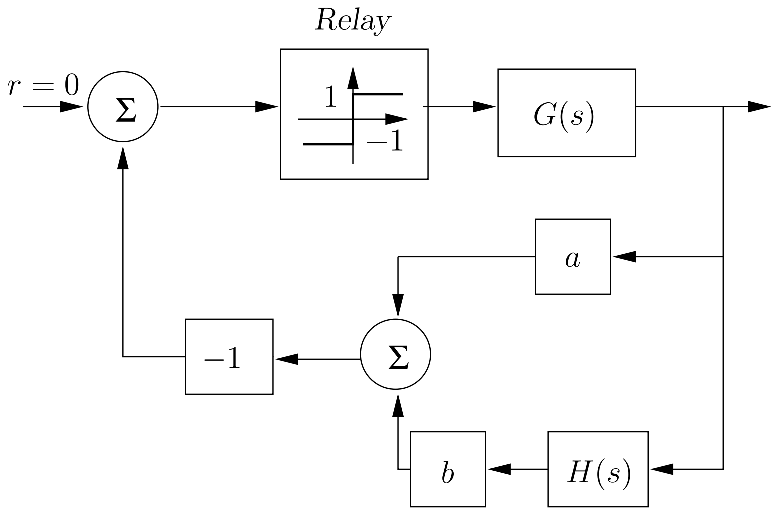
$$\ell = \ell_0 + \varepsilon\Phi\dot{\Phi}, \qquad \varepsilon > 0$$

will bring the swing to rest in $\Phi = 0$.

Solution

12.7

The block diagram below is given.



We have that

$$H(s) = s \qquad \text{and} \qquad G(s) = \frac{1}{(s+1)(s+2)}.$$

How shall the feedback coefficients a and b be chosen to guarantee Lyapunov stability?

Hint: Use a quadratic Lyapunov function candidate.

Solution

12.8

A servo system contains a nonlinearity where the relationship between the input signal u and the output signal y is

$$y = u + \arctan(u)$$

What requirements on the linear part of the servo system must be fulfilled in order to prove stability using the circle criterion?

Solution

12.9

A simplified model for the movements of an airplane is given by

$$\dot{x} = \begin{bmatrix} -0.01 & 0.03 & -10 \\ 0 & -1 & 300 \\ 0 & 0 & -0.5 \end{bmatrix} x + \begin{bmatrix} 4 \\ -20 \\ -10 \end{bmatrix} u$$

All states are measured and the control signal is

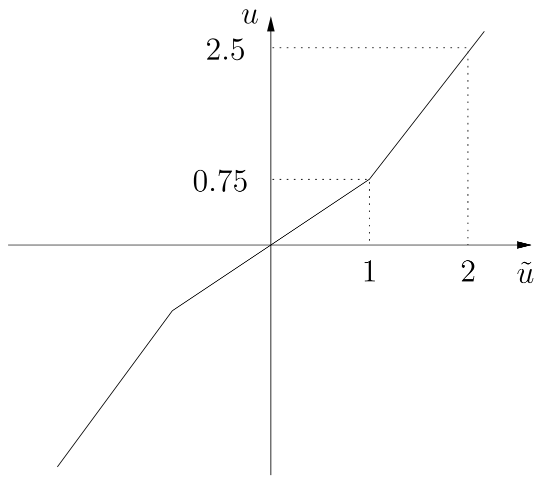
$$\tilde{u} = -Lx + \tilde{r}$$

where L is the feedback that minimizes

$$\int (x^T(t) Q_1 x(t) + u^2(t))\, dt$$

for $Q_1 = 10 \cdot I$

The requested control signal $\tilde{u}$ is different from the actual u, due to the hydraulic servo dynamics. The relationship between $\tilde{u}$ and u is



Will the closed-loop system be stable? Motivate your answer.

Solution

13 Phase Plane Analysis

13.1

Given the differential equation

$$\ddot{y} - (0.1 - \frac{10}{3}\dot{y}^2)\dot{y} + y + y^2 = 0$$

(a) Convert the model to a nonlinear state-space model.

(b) Find the stationary points.

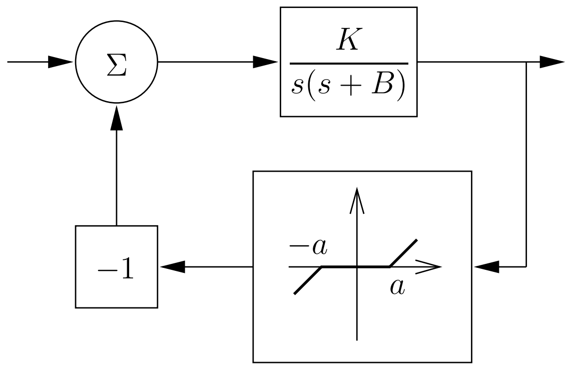
(c) Compute linearizations around the stationary points and analyze stability.

(d) Draw the phase portrait of this system.

Solution

13.2

Draw the phase portrait of the depicted position servo.

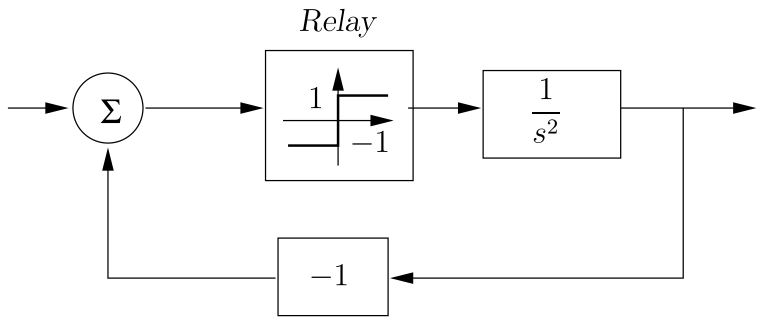


The position is measured using an E-transformer, which can be described as a dead zone. Assume that $K > \frac{B^2}{4}$.

Solution

13.3

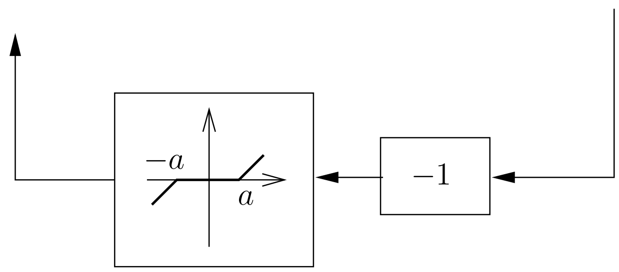
The following system is given



(a) With zero input signal the output of the relay is +1 or −1, depending on the history of the input signal. The relay does not switch until the input signal has changed polarity.

Draw a phase portrait of the system.

(b) Due to imperfections the actual feedback loop is

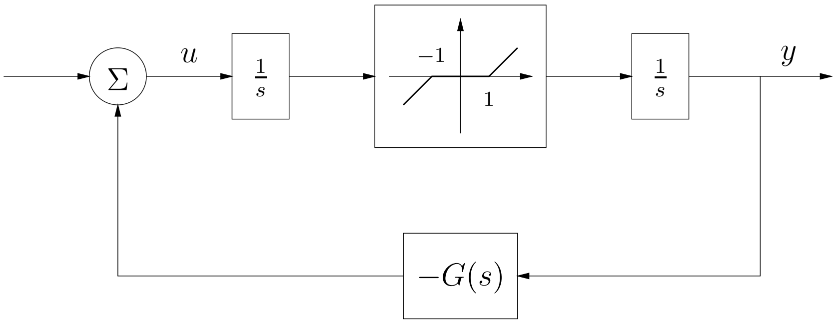


Draw the phase portrait of this system.

Solution

13.4

Linus is on his way home after an exam. On the highway outside of Linkï¿½ping a gust of wind makes the car drift from the desired path. Your task is to, using phase plane analysis, decide how the movement of the car will progress. Will it return to the desired path? If the car has a constant speed in the direction of travel the system can be described by the following block diagram



The torque applied to the steering wheel is u. The backlash comes from a gear unit in the steering. The output signal y is the deviation from the desired path. $G(s)$ is the transfer function from Linus' visual perception to the torque he applies to the steering wheel.

Distinguish between the cases:

(a) $G(s) = 1$ (there was a party after the exam)

(b) $G(s) = 1 + s$ (there was not a party after the exam)

Solution

13.5

A simple ecological system consists of two species of fish. The first kind eats algae and the second kind eats the first kind. Let x_1 denote the number of algae eating fish and x_2 denote the number of predatory fish. Then we have

$$\dot{x}_1 = 2x_1 - \frac{x_1 x_2}{1 + \frac{1}{6}x_1} - 0.2x_1^2$$
$$\dot{x}_2 = -3x_2 + \frac{x_1 x_2}{1 + \frac{1}{6}x_1}$$

(a) From these equations, calculate the stationary points.

(b) Classify the stationary points and sketch the phase portraits in a surrounding of them. It is sufficient to consider a linearised version of the equations.

(c) Without any further calculations, merge the phase portraits you have made around the stationary points in a fashion that seems reasonable. Only consider $x_1 > x_2 > 0$.

An interpretation of the given equations is:

If the algae eating fish have an infinite amount of food and lack enemies, their number will grow exponentially as

$$\dot{x}_1 = 2x_1$$

As there is a limited amount of algae the growth saturates according to

$$\dot{x}_1 = 2x_1 - 0.2x_1^2.$$

If there are predatory fish x_2 present the algae eaters will be devoured at the rate

$$\frac{x_1 x_2}{1 + \frac{1}{6}x_1}$$

The interpretation of this term is that if x_1 is large every predatory fish can eat until it is full. This corresponds to 6 algae eating fish per time unit. On the other hand, if the number x_1 is relatively small the predatory fish will eat less.

The second equation says that if the supply of food is unlimited ($x_1 = \infty$) the predatory fish will multiply according to

$$\dot{x}_2 = 3x_2.$$

If food is lacking ($x_1 = 0$) the predatory fish will expire as

$$\dot{x}_2 = -3x_2$$

Solution

13.6

A mass is suspended from a spring. Its position $y(t)$ satisfies the differential equation

$$\ddot{y}(t) + y(t) = f(t)$$

where $f(t)$ is an external force acting on the mass. Draw a phase portrait of the system when

$$f(t) = \begin{cases} -1 & \text{if } \dot{y}(t) > 0 \\ +1 & \text{if } \dot{y}(t) < 0 \end{cases}$$

Will the system reach an equilibrium?

Solution

13.7

Consider the system

$$\dot{x} = \begin{pmatrix} -x_1^3 + u \\ x_1 \end{pmatrix}$$

(a) Sketch a phase portrait when $u = 0$.

(b) Use the Lyapunov function

$$V(x) = x_1^2 + x_2^2$$

to compute a control signal

$$u = f(x_1, x_2)$$

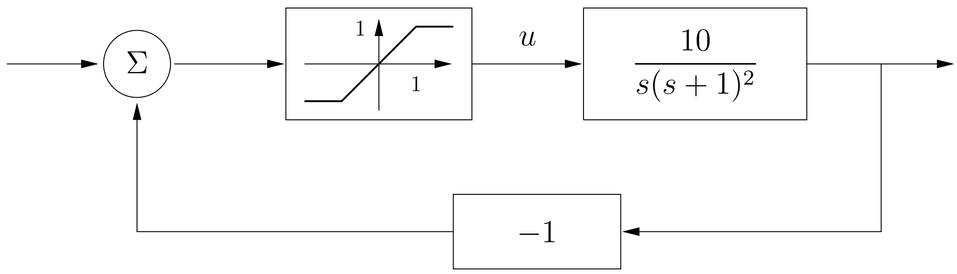
which will make the origin globally asymptotically stable. Sketch a phase portrait, in a neighborhood of the origin, for the closed-loop system.

Solution

14 Oscillations and Describing Functions

14.1

Consider the feedback control system including an input saturation according to the figure below.



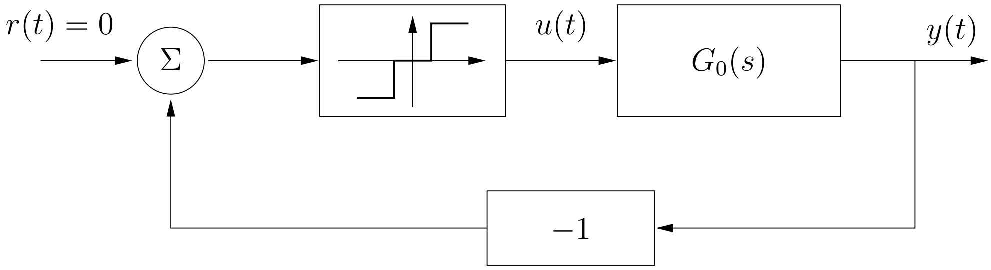
(a) Investigate the stability of the system. If a periodical solution exists, determine its frequency and amplitude.

(b) Build a simulation model of the control system and investigate the validity of the results from a).

Solution

14.2

A temperature control system, depicted below, contains a relay with dead zone.



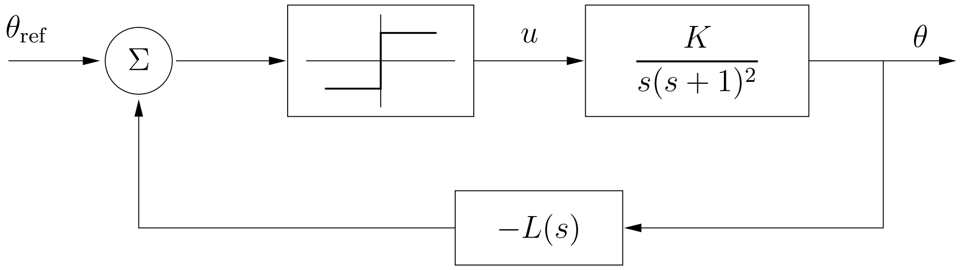
$G_0(s) = \frac{1}{s(1+s)^2}$, $\pm D$ is the width of the dead zone and $\pm H$ is the output level of the relay. The values of the dead zone and output level are such that a stable oscillation just barely can exist. If H is increased or if D is decreased an oscillation will not be possible.The amplitude of the oscillation is 2.5 units. Compute D, H and the frequency of the oscillation. The describing function for a relay with dead zone is

$$\begin{aligned} \mathrm{Re}\{Y_N(C)\} &= \frac{4H}{\pi C}\sqrt{1 - D^2/C^2}, \qquad C \geq D \\ \mathrm{Im}\{Y_N(C)\} &\equiv 0 \end{aligned}$$

Solution

14.3

A relay servo is given by



The gain K is strictly positive.

(a) The feedback used is $L(s) = 1$. Show that there is an oscillation for all values of K.

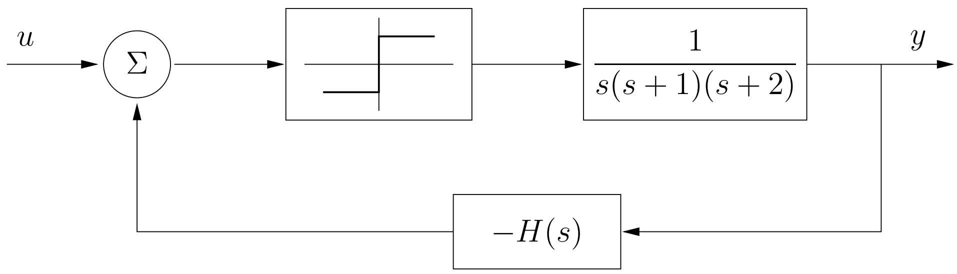
(b) To avoid too much wear on the system we do not want the amplitude of the oscillation in θ to be greater than 0.1. For what values of K is this fulfilled?

(c) We want to use a gain K that is larger than what is possible in (b). State a feedback $L(s)$ with $L(0) = 1$ that makes this feasible. No details are necessary. Just motivate why the feedback should solve the problem.

Solution

14.4

Consider the nonlinear system



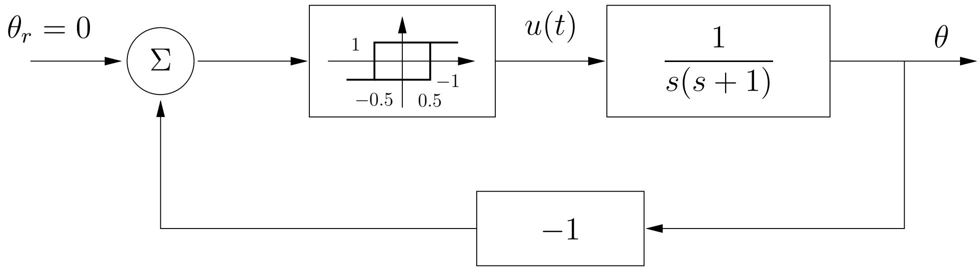
(a) If proportional control is used, i.e. $H(s) = 1$, a stable oscillation occurs. Find the amplitude and frequency of the oscillation.

(b) To eliminate the oscillation we use proportional and derivative control, i.e. $H(s) = 1 + Ks$. Show how K can be chosen to eliminate the oscillation.

Solution

14.5

Consider the feedback control system where a motor is controlled using a relay with hysteresis.



(a) Investigate the stability of the system using the describing function method. If a periodical solution exists, determine its frequency and amplitude.

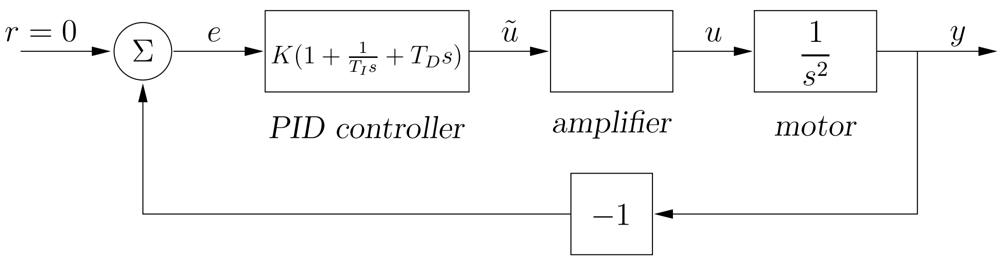
(b) Build a simulation model of the control system and investigate the validity of the results from a).

(c) Introduce suitable state variables and sketch a phase portrait.

Solution

14.6

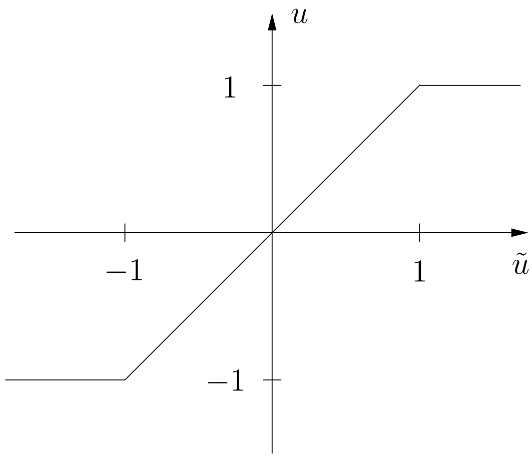
Consider the following servo system



The PID controller has $K = 2$, $T_I = 2$ and $T_D = 0.5$.

(a) The tuning of the controller was done assuming that the amplifier has the transfer function 1. Show that, if this assumption is true, this results in an asymptotically stable closed-loop system.

(b) The actual amplifier contains a saturation



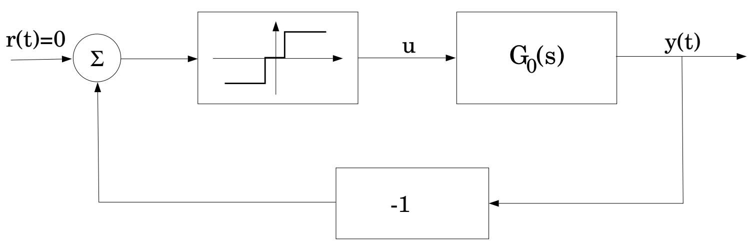
State the amplitude, frequency and stability properties of possible oscillations.

(c) Discuss, based on the results from (b), under what circumstances the servo system will function as intended. Especially investigate the influence of different signal amplitudes.

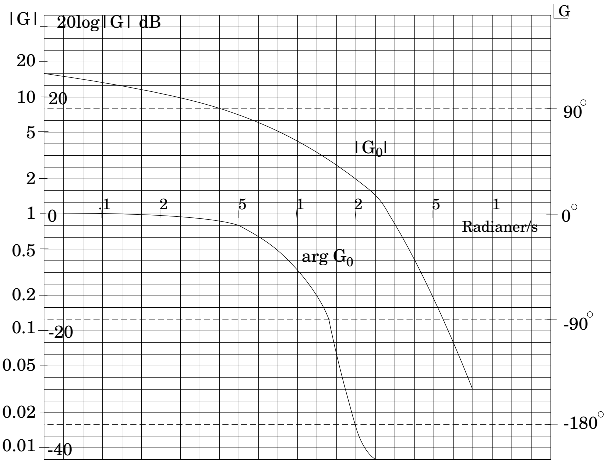
Solution

14.7

Z Consider the feedback control system



When the system is simulated a limit cycle occurs. Determine the amplitude and frequency of the limit cycle. The Bode diagram for the linear part $G_O(s)$ of the control system is given in the figure below.



The describing function of a relay with deadzone is given by

$$Y_f(C) = \frac{4}{\pi C}\sqrt{1 - 1/C^2} \quad C \geq 1$$

Solution

17 To Compensate Exactly for Nonlinearities

17.1

Find a feedback which makes the system

$$\begin{aligned}\dot{x}_1 &= -x_1 + 7x_2\\ \dot{x}_2 &= -x_2 + \cos x_1 + u\end{aligned}$$

linear.

Solution

17.2

Find an output feedback, $u = f(y)$, which makes the system

$$\begin{aligned}\dot{x}_1 &= x_3 + 8x_2\\ \dot{x}_2 &= -x_2 + x_3\\ \dot{x}_3 &= -x_3 + x_1^4 - x_1^2 + u\\ y &= x_1\end{aligned}$$

linear.

Solution

17.3

Find a feedback which makes the system

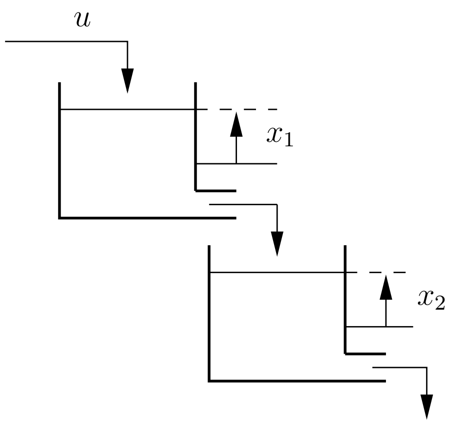
$$\begin{aligned}\dot{x}_1 &= x_1^2 + x_2\\ \dot{x}_2 &= u\\ y &= x_1\end{aligned}$$

linear.

Solution

17.4

Consider the two tank system:



The dynamics of the system is described by

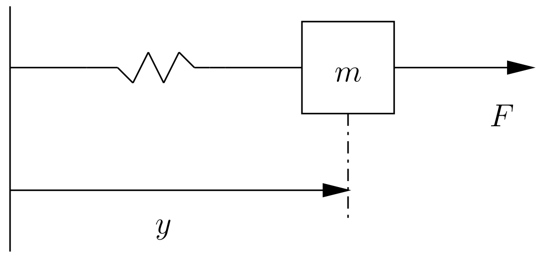
$$
\begin{aligned}
\dot{x}_1 &= 1 + u - \sqrt{1 + x_1} \\
\dot{x}_2 &= \sqrt{1 + x_1} - \sqrt{1 + x_2}
\end{aligned}
$$

Which state should be chosen as output to achieve a strong relative degree 2? Do a feedback linearization of the system.

Solution

17.5

A mass m is suspended from a spring:



The force F is generated by the control signal u fed through an actuator such that

$$F = \frac{1}{s+1}u$$

The position of the mass is y. The spring rate and the viscous damping are nonlinear. Thus the force is

$$-k(y) - d(\dot{y}).$$

(a) Realize this system on state-space form. The input signal is u and the output signal is y.

(b) Can the system in (a) be made linear using feedback? If so, compute such a feedback.

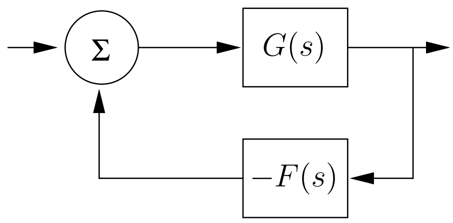
Solution

Solutions

1 Introduction

1.1

The small gain theorem for linear systems can be stated as follows: Assume that both $G(s)$ and $F(s)$ are stable transfer functions, and interconnected according to the figure below.



Then the closed-loop system is stable if

$$|G(i\omega)| \cdot |F(i\omega)| < 1, \quad \forall\omega.$$

The transfer function of the closed-loop system is

$$G_c(s) = \frac{G(s)}{1 + G(s)F(s)}$$

$G_c(s)$ is stable according to the Nyquist criterion if the Nyquist curve for $G(i\omega)F(i\omega)$ does *not* encircle the point -1. Since we know from the small gain theorem that

$$|G(i\omega)F(i\omega)| \leq |G(i\omega)| \cdot |F(i\omega)| < 1,$$

the Nyquist curve can not encircle the point -1, and hence the Nyquist criterion is fulfilled.

Note that input-output stability follows from asymptotic stability. Input-output stability is the concept used in the general small gain theorem.

Go back

1.2

We have that $y(t) = f(u(t))$ where $f(\cdot)$ is the function describing the ideal relay. The gain is defined as

$$\|f\| = \sup_{u \neq 0} \frac{\|y\|_2}{\|u\|_2}$$

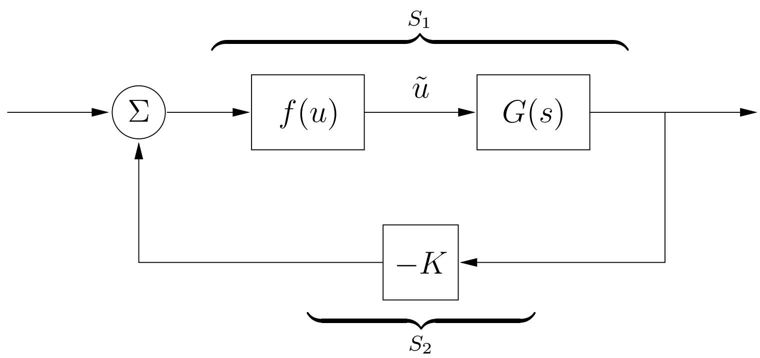
We have that $|f(u)| \equiv 1, \ \forall u(t) \neq 0$, and this yields

$$\|y\|_2^2 = \int_{-\infty}^{\infty} y^2(t)dt = \lim_{T\to\infty} \int_{-T}^{T} [f(u(t))]^2 dt = \lim_{T\to\infty} 2T = \infty$$

for *all* choices of $u(t) \neq 0$ such that $0 < \|u\|_2 < \infty$. Take for example $u(t) = \frac{1}{t}$. This means that an ideal relay has *infinite* gain.

Go back

1.3



The system is stable according to the small gain theorem if $\|S_1\| \cdot \|S_2\| < 1$.

We have that: $\begin{cases} \|S_1\| \leq \|f(u)\| \cdot \|G\| \\ \|S_2\| = |K| \end{cases}$

where

$$\|G\| = \sup_{\omega} |G(i\omega)| = \sup_{\omega} \frac{2}{\sqrt{(2-\omega^2)^2 + 4\omega^2}} = 1 \quad (\text{fi¿½r } \omega = 0)$$

$$\|f(u)\|^2 = \frac{\|\tilde{u}\|_2^2}{\|u\|_2^2} = \frac{\int_{-\infty}^{\infty}(f(u(t))^2dt}{\|u\|_2^2} \leq \left[|f(u(t))| \leq \frac{1}{2}|u(t)|\right]$$
$$\leq \frac{\frac{1}{4}\|u\|_2^2}{\|u\|_2^2} = \frac{1}{4} \quad \Rightarrow \quad \|f(u)\| \leq \frac{1}{2}$$

$$\|S_1\| \cdot \|S_2\| \leq \frac{1}{2} \cdot |K| < 1$$

i.e. , we must choose $|K| < 2$ to be able to guarantee input-output stability.

Go back

1.4

(a) $\|y\|_\infty = |a|, \qquad \|y\|_2 = \infty$

(b) $\|y\|_\infty = 1, \qquad \|y\|_2 = 1$

(c) $\|y\|_\infty = \frac{1}{4}, \qquad \|y\|_2 = \frac{1}{6}\sqrt{3}$

Go back

1.5

The gain of the system is

$$\|G\| = \sup_\omega |G(i\omega)| = \sup_\omega \frac{\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\zeta^2\omega_0^2\omega^2}}$$

By differentiating $|G(i\omega)|$ we see that the magnitude of $G(i\omega)$ has its maximum at $\omega = 0$ if $\zeta > \frac{1}{\sqrt{2}}$. This results in the gain

$$\|G\| = 1$$

If $0 < \zeta < \frac{1}{\sqrt{2}}$ the maximum of $|G(i\omega)|$ is attained at $\omega = \omega_0\sqrt{1-2\zeta^2}$. This results in the gain

$$\|G\| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

Go back

1.6

One has to distinguish between the cases $a > 0$ and $a < 0$ respectively.

(i) For $a > 0$ the system $G(s)$ is stable and the small gain theorem is applicable. The system $G(s)$ has gain one, and the small gain theorem hence gives the condition $\mid K \mid < 1$. The characteristic equation of the closed loop system is given by (Note: positive feedback)

$$(s + a) - Ka = 0$$

which imples the pole $s = (K - 1)a$, which is located in the left half plane for $K < 1$.

(ii) For $a < 0$ the system $G(s)$ is not stable and the small gain theorem is not applicable. The pole $s = (K - 1)a$ is in the left half plane for $K > 1$.

Go back

1.7

The linear part, represented by $G(s)$, has gain 1.5 according to the figure. For the nonlinear part we assume that $f(x)$ is an odd function, such that $f(-x) = -f(x)$. The nonlinearity can hence be bounded by

$$\mid f(x) \mid \leq 0.5 \mid x \mid$$

and hence the gain is 0.5. Since $1.5 \cdot 0.5 < 1$ the closed loop system is stable according to the small gain theorem.

Go back

1.8

Using proportional control $u = -Ky = -Kx_1$ we get

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -aKx_1 - ax_2\end{aligned}$$

The characteristic equation is $\quad s^2 + as + aK = s^2 + (1+\rho)s + (1+\rho)K = 0$
The closed-loop system is stable if $(1 + \rho) > 0$ and $(1 + \rho)K > 0$, i.e. it is stable for all $K > 0$ when $\delta < 1$.

Go back

1.9

(a) With $a = 1 + \rho$ it holds that

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a(Kx_1 + x_2) = -Kx_1 - x_2 - \underbrace{\rho(Kx_1 + x_2)}_{w=\rho z}\end{aligned}$$

The open system with input signal w and output signal z is given by

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 0 & 1 \\ -K & -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ -1 \end{pmatrix} w \\ z &= \begin{pmatrix} K & 1 \end{pmatrix} x,\end{aligned}$$

that is

$$G_{wz}(s) = -\frac{s+K}{s^2+s+K}.$$

(b) The relationship

$$w(t) = \rho(t)z(t)$$

gives

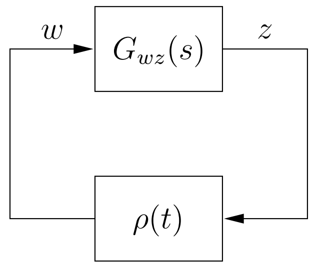
$$\|w(t)\|_2^2 = \int_{\infty}^{\infty} \rho^2(t)z^2(t)dt < \delta^2\|z(t)\|_2^2$$

and

$$\frac{\|w(t)\|_2}{\|z(t)\|_2} < \delta$$

which implies that the gain is at most δ.

(c) The closed-loop system is depicted in the figure below.



A sufficient condition for stability, according to the low gain theorem, is

$$\|G_{wz}\| \cdot \|\rho\| < 1 \quad \Leftrightarrow \quad \|G_{wz}\|^2 < 1/\delta^2$$

The magnitude, of the linear systems frequency response, squared is

$$|G_{wz}(i\omega)|^2 = \frac{\omega^2 + K^2}{(K-\omega^2)^2 + \omega^2}$$

What is the maximum of this function?

$$\sup_x \frac{x + K^2}{(K-x)^2 + x} = \frac{\sqrt{K^4 + 2K^3}}{2K^4 + 4K^3 - (2K + 2K^2 - 1)\sqrt{K^4 + 2K^3}}$$

for $x = -K^2 + \sqrt{K^4 + 2K^3}$. Thus, an implicit condition on K to guarantee stability of the closed-loop system is

$$\frac{\sqrt{K^4 + 2K^3}}{2K^4 + 4K^3 - (2K + 2K^2 - 1)\sqrt{K^4 + 2K^3}} < 1/\delta^2$$

Go back

2 Representation of Linear Systems

2.1

Putting $x = \omega, u_1 = M, u_2 = I_m, u_3 = R, y_1 = \omega$ and $y_2 = e$ give the state equation

$$\dot{x} = u_1 - \frac{x^2 u_2}{u_3}$$

and

$$y_1 = x$$

$$y_2 = u_2 x$$

The input vector $u_{1,0} = u_{2,0} = u_{2,0} = 1$ and the state x_0 is a stationary point, giving the stationary output $y_{1,0} = y_{2,0} = 1$. Introducing

$$\Delta x = x - x_0 \quad \Delta u = u - u_0 \quad \Delta y = y - y_0$$

and linarizing gives

$$\frac{d}{dt}\Delta x = -2\Delta x + \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \Delta u$$

$$\Delta y = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Delta x + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \Delta u$$

Computing the transfer function and renaming the variables give

$$\begin{pmatrix} \Delta\omega \\ \Delta e \end{pmatrix} = \frac{1}{s+2} \begin{pmatrix} 1 & -1 & 1 \\ 1 & s+1 & 1 \end{pmatrix} \begin{pmatrix} \Delta M \\ \Delta I_m \\ \Delta R \end{pmatrix}$$

Go back

2.2

(a)

$$y = \alpha h_2, \qquad f = \beta(h_1 - h_2)$$
$$\dot{h}_1 = \frac{1}{A_1}(u_1 - f), \qquad \dot{h}_2 = \frac{1}{A_2}(u_2 + f - y)$$

$$\dot{h} = \begin{pmatrix} -\frac{1}{A_1}\beta & \frac{1}{A_1}\beta \\ \frac{1}{A_2}\beta & -\frac{1}{A_2}(\beta+\alpha) \end{pmatrix} h + \begin{pmatrix} \frac{1}{A_1} & 0 \\ 0 & \frac{1}{A_2} \end{pmatrix} u$$
$$y = \begin{pmatrix} 0 & \alpha \end{pmatrix} h$$

$$G(s) = \frac{1}{s^2 + (2\beta+\alpha)s + \alpha\beta} \begin{pmatrix} \alpha\beta & \alpha(s+\beta) \end{pmatrix}$$

(b) The result above gives

$$G(0) = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

Singular values in $\omega = 0$ (i.e., for constant input signals) are given by the square roots of the largest and smallest eigenvalues of the matrix

$$G(0)^T G(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Solving $\det(\lambda I - G(0)^T G(0)) = 0$ yields eigenvalues 0 and 2, which implies

$$\bar{\sigma}(G(0)) = \sqrt{2} \qquad \underline{\sigma}(G(0)) = 0$$

The maximum singluar value corresponds the input signal vector

$$u_{max} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

which gives the steady state output signal $y = 2$, while the minumim singular value corresponds the input signal vector

$$u_{min} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

which gives the steady state output signal $y = 0$.

Go back

2.3

Find the common denominator of the system

$$\begin{aligned} Y(s) &= \frac{(s^2+s+1)}{(s+1)(s+2)(s^2+s+1)} U_1(s) + \frac{(s+2)(s+3)}{(s+1)(s+2)(s^2+s+1)} U_2(s) \\ &= \frac{(s^2+s+1)}{(s^4+4s^3+6s^2+5s+2)} U_1(s) + \frac{(s^2+5s+6)}{(s^4+4s^3+6s^2+5s+2)} U_2(s) \end{aligned}$$

It is now straightforward to realize the system on observer canonical form

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} -4 & 1 & 0 & 0 \\ -6 & 0 & 1 & 0 \\ -5 & 0 & 0 & 1 \\ -2 & 0 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 5 \\ 1 & 6 \end{pmatrix} u(t) \\ y(t) &= (1 \quad 0 \quad 0 \quad 0)\, x(t) \end{aligned}$$

Go back

2.4

Find the common denominator

$$y(t) = \frac{1}{p^3+7p^2+16p+12} \begin{pmatrix} p^2+3p & p^2+p-2 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

Observer canonical form yields

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} -7 & 1 & 0 \\ -16 & 0 & 1 \\ -12 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 & 1 \\ 3 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \\ y(t) &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x(t) \end{aligned}$$

Go back

2.5

Laplace transformation yields

$$Y(s) = \frac{(b_{11}s + b_{12})}{(s^2 + a_1 s + a_2)} U_1(s) + \frac{(b_{21}s + b_{22})}{(s^2 + a_1 s + a_2)} U_2(s)$$

The system on observer canonical form is

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} -a_1 & 1 \\ -a_2 & 0 \end{pmatrix} x(t) + \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix} u(t) \\ y(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x(t) \end{aligned}$$

Go back

2.6

Laplace transformation yields

$$A(s)Y(s) = B(s)U(s)$$

where

$$A(s) = \begin{pmatrix} s & 1 \\ 1 & (s+1) \end{pmatrix} \qquad B(s) = \begin{pmatrix} s+2 \\ 1 \end{pmatrix}$$

Multiplication by $A^{-1}(s)$ results in

$$Y(s) = A^{-1}(s)B(s)U(s)$$

i.e.

$$Y(s) = \begin{pmatrix} \frac{s^2+3s+1}{s^2+s-1} \\ \frac{-2}{s^2+s-1} \end{pmatrix} U(s) = \begin{pmatrix} \frac{2s+2}{s^2+s-1} + 1 \\ \frac{-2}{s^2+s-1} \end{pmatrix} U(s)$$

The system on controller canonical form

$$\dot{x}(t) = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t)$$

$$y(t) = \begin{pmatrix} 2 & 2 \\ 0 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t)$$

Go back

3 Properties of Linear Systems

3.1

The transfer function matrix has the minors

$$-\frac{1}{(s+2)^2} - \frac{(s+1)}{(s+2)^2} = -\frac{1}{s+2}$$

when the first column is deleted,

$$\frac{1}{(s+2)^2} - \frac{1}{(s+2)^2} = 0$$

when the second column is deleted and

$$\frac{(s+1)}{(s+2)^2} + \frac{1}{(s+2)^2} = \frac{1}{(s+2)}$$

when the third column is deleted. In addition, the elements of the transfer function are themselves minors. The pole polynomial, i.e. the least common denominator to all minors is thus

$$p(s) = (s+2)$$

The system has a pole in $s = -2$. Hence, the system can be realized as a state-space sytem of order one.

The maximal minors are

$$-\frac{1}{s+2}, \qquad 0, \qquad \frac{1}{(s+2)}$$

Thus, the zero polynomial is a constant. The system lacks zeros.

Go back

3.2

The transfer function matrix has the determinant

$$\det G(s) = \frac{2}{(s+3)^2}$$

and the minors

$$\frac{1}{(s+1)(s+3)}, \qquad \frac{-1}{(s+1)(s+3)}, \qquad \frac{2(s+1)}{(s+3)}$$

The pole polynomial, i.e. the least common denominator of the minors, is

$$p(s) = (s+1)(s+3)^2,$$

Hence the poles are $-1, -3$ and -3. The maximal minor is

$$\frac{2}{(s+3)^2}$$

If we normalize with the pole polynomial we get

$$\frac{2(s+1)}{(s+1)(s+3)^2}.$$

The zero polynomial is thus

$$n(s) = (s+1)$$

There is a zero in -1.

Go back

3.3

Minors:

$$\underbrace{\frac{1-s}{(s+1)^2}}_{2\text{ st}}, \quad \frac{2-s}{(s+1)^2}, \quad \frac{1/3-s}{(s+1)^2}, \quad \frac{1/3}{(s+1)^3}$$

There are 3 poles in $s = -1$ as the least common denominator is $(s+1)^3$. Thus a minimal realization must be of order three.

Go back

3.4

(a) The determinant of the transfer function matrix is

$$\frac{(s+5)}{(s^2+3s+2)(s+2)} - \frac{1}{(s+2)(s+4)} = \frac{6(s+3)}{(s+1)(s+2)(s+2)(s+4)}$$

and the minors are

$$\frac{(s+5)}{s^2+3s+2}, \quad \frac{1}{(s+2)}, \quad \frac{1}{(s+4)}, \quad \frac{1}{(s+2)}$$

Thus, the pole polynomial is

$$p(s) = (s+1)(s+2)(s+2)(s+4)$$

which means that the poles are located at $-1, -2, -2$ and -4. We need four states to realize the system.

(b) The determinant of the transfer function matrix is

$$\frac{(s+5)}{(s+4)(s^2+3s+2)} - \frac{1}{(s+2)(s+4)} = \frac{4}{(s+1)(s+2)(s+4)}$$

The pole polynomial is

$$p(s) = (s+1)(s+2)(s+4)$$

which means that the poles are located at $-1,\ -2$ and -4. We need three states to realize the system.

Go back

3.5

The system can be written on the form

$$A(p)y(t) = B(p)u(t)$$

where

$$A(p) = \begin{pmatrix} (p+1) & -p \\ p & (p+1) \end{pmatrix} \qquad B(p) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Multiplication with $A^{-1}(p)$ yields

$$G(p) = A^{-1}(p)B(p) = \frac{1}{2p^2 + 2p + 1} \begin{pmatrix} (2p+1) & -1 \\ 1 & (2p+1) \end{pmatrix}$$

The transfer function matrix, $G(s)$, has the determinant

$$\det G(s) = \frac{(2s+1)^2}{(2s^2+2s+1)^2} + \frac{1}{(2s^2+2s+1)^2} = \frac{2}{(2s^2+2s+1)}$$

and the minors

$$\frac{(2s+1)}{(2s^2+2s+1)} \qquad \frac{-1}{(2s^2+2s+1)} \qquad \frac{1}{(2s^2+2s+1)}$$

This results in the pole polynomial

$$p(s) = 2s^2 + 2s + 1$$

Hence, the poles are located at $-\frac{1}{2} \pm i\frac{1}{2}$.

The maximal minor is

$$\frac{2}{(2s^2+2s+1)}$$

Thus, there are no zeros of the system.

Go back

3.6

Alt. 1: The output signal y only depends on x_1 and x_2. The states x_1, x_2 do not depend on x_3 due to the structure of the matrix A. Hence, the state x_3 is unobservable and can be eliminated from the state-space form:

$$\dot{\tilde{x}}(t) = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix} \tilde{x}(t) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u(t), \quad y(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tilde{x}(t),$$

where $\tilde{x} = (x_1, x_2)^T$. The controllability matrix and the observability matrix both have full rank and hence the realization is minimal.

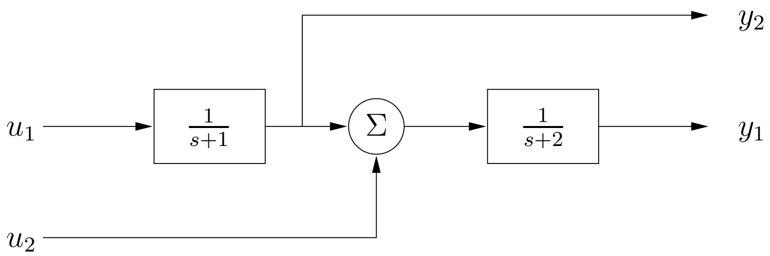
Alt. 2: The transfer function matrix can be computed as $G(s) = C(sI - A)^{-1}B$

$$G(s) = \begin{pmatrix} \frac{1}{(s+2)(s+1)} & \frac{1}{s+2} \\ \frac{1}{s+1} & 0 \end{pmatrix},$$

i.e.

$$y_1 = \frac{1}{(s+2)(s+1)}u_1 + \frac{1}{s+2}u_2$$
$$y_2 = \frac{1}{s+1}u_1$$

This results in the block diagram:



Introduce a state after each block, for example $x_1 = y_1$ and $x_2 = y_2$. This results in the same minimal realization as in Alt.1.

Go back

3.7

(a) The singular values at $\omega = 2$ can be determined in two ways, and for both alternatives we start by entering the transfer function matrix in Matlab.

```
>> s=tf('s');
>> G=[1/(s+1) 3/(s+2); 2/(s+3) 1/(s+4)];
```

Alternative (i): The frequency function $G(i\omega)$ at the angular frequency $\omega = 2$ can be computed according to

```
>> G2 = freqresp(G,2)

G2 =

   0.2000 - 0.4000i   0.7500 - 0.7500i
   0.4615 - 0.3077i   0.2000 - 0.1000i
```

The eigenvalues and eigenvector of $G(i\omega)^*G(i\omega)$ are now obtained from

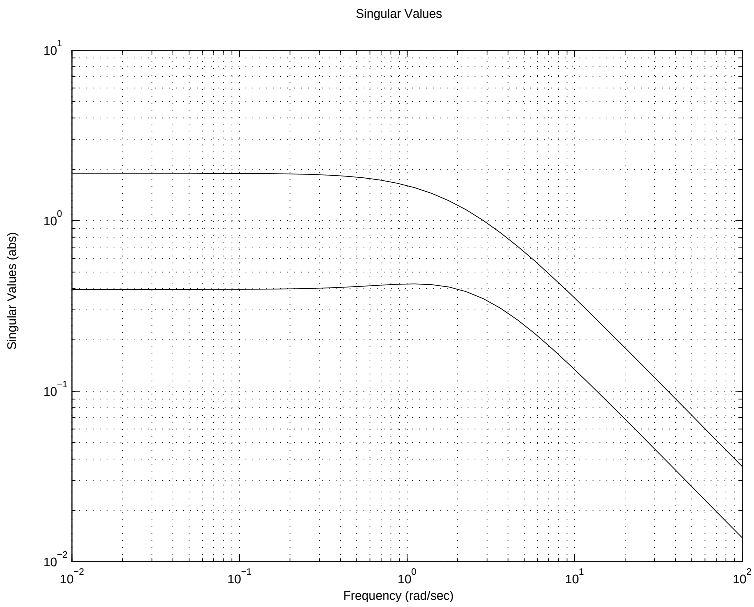
```
>> [V,D]=eig(G2'*G2)

V =

   0.8288 + 0.2392i   0.4860 + 0.1403i
  -0.5059             0.8626


D =

    0.1579         0
         0    1.5248
```

The smallest singular value is hence $\underline{\sigma}(G(i2)) = \sqrt{0.1579} \approx 0.40$ and the largest is $\bar{\sigma}(G(i2)) = \sqrt{1.5248} \approx 1.24$.

Alternative (ii): The singular values can be detemined graphically using the command

```
>> sigma(G)
```

which gives the result



(b) The second column of the matrix `V` defines the Fourier transform of the input vector that corresponds to the largest gain of the system, i.e. the input vector is such that the input components fulfill

$$| U_1(i\omega) |= \sqrt{0.486^2 + 0.1403^2} \approx 0.51$$

$$\arg U_1(i\omega) = \arctan(0.1403/0.486) \approx 0.28 \;\; rad$$

and

$$| U_2(i\omega) |\approx 0.86 \qquad \arg U_2(i\omega) = 0$$

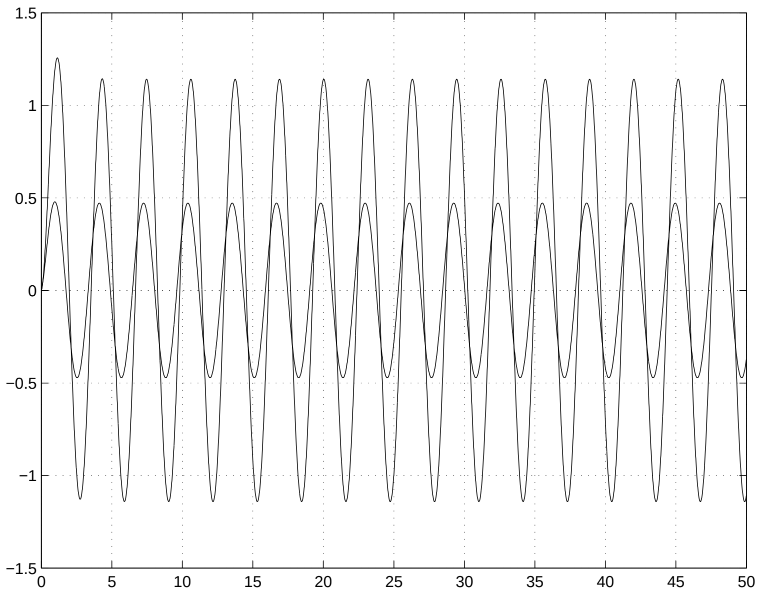
(c) Using the hints an input vector can be generated using the command sequence

```
>> t=(0:0.01:50).';
>> u12=0.51*sin(2*t+0.28);
>> u22=0.86*sin(2*t);
```

The system can then be simulated using

```
>> y2=lsim(G,[u12 u22],t);
>> plot(t,y2)
```

This gives the result below showing that the output components have amplitudes 1.14 and 0.48.



(d) Using the fact that the Fourier transform at the studied frequency are proportional to the signal amplitude the ratio of the norms of the output and input vectors becomes

$$\frac{\sqrt{1.14^2 + 0.48^2}}{\sqrt{0.51^2 + 0.86^2}} \approx 1.24$$

which corresponds to the largest singular value.

Go back

3.8

The minors of order 1 are

$$\frac{1}{s+1},\ \frac{s-1}{(s+1)(s+2)},\ \frac{-1}{s-1},\ \frac{1}{s+2}$$

The minors of order 2 are

$$\frac{-(s-1)}{(s+1)(s+2)^2},\ \frac{2}{(s+1)(s+2)},\ \frac{1}{(s+1)(s+2)}.$$

The least common denominator yields the pole polynomial

$$p(s) = (s+1)(s+2)^2(s-1),$$

and the poles are therefore $-1,\ -2,\ -2,\ 1$. The maximal minors, normalized with the pole polynomial, are then given by

$$\frac{-(s-1)^2}{(s+1)(s+2)^2(s-1)}, \quad \frac{2(s-1)(s+2)}{(s+1)(s+2)^2(s-1)}, \quad \frac{(s-1)(s+2)}{(s+1)(s+2)^2(s-1)},$$

and the gcd of the numerators is thus $z(s) = s - 1$ and the only zero is 1.

Go back

5 Disturbance Models

5.1

$\Phi_u(\omega)$ is an even function. Do the decomposition $\Phi_u(\omega) = G(i\omega)G(-i\omega)\Phi_e(\omega)$ where $G(s)$ has all poles and zeros in the left-half plane and $\Phi_e = 1$.

(a)
$$\Phi_u(\omega) = \frac{a^2}{\omega^2 + a^2}\Phi_e(\omega) = \frac{a}{i\omega + |a|} \cdot \frac{a}{-i\omega + |a|}$$

Thus the linear filter is

$$G(s) = \frac{a}{s + |a|}, \qquad a \neq 0.$$

(b) Analogously we get

$$\Phi_u(\omega) = \frac{a^2 b^2}{(\omega^2 + a^2)(\omega^2 + b^2)}\Phi_e(\omega)$$
$$= \frac{ab}{(i\omega + |a|)(i\omega + |b|)} \cdot \frac{ab}{(-i\omega + |a|)(-i\omega + |b|)}$$
$$\Rightarrow \quad G(s) = \frac{ab}{(s + |a|)(s + |b|)}$$

Go back

5.2

Consider the disturbance model

$$N(s) = H(s)V(s)$$

where V denotes white noise. In (i) the transfer function is of low pass character, which means that N will be of low frequency character. The disturbance is located around 5 Hz, i.e. 10π rad/s. The magnitude curve of model (ii) has a peak around this angular frequency, which means that this model is the most appropriate one. In model (iii) the peak is located around 5 rad/s.

Go back

5.3

(a) We are given

$$f = k_1\dot{z} + v$$

The force is $m\ddot{z} = u - f$, where m is the mass of the missile and u is the thrust.

On input-output form:

$$\ddot{z} + \frac{k_1}{m}\dot{z} = \frac{1}{m}(u - v)$$

State-space form: Let $x_1 = z$, $x_2 = \dot{z} \quad \Rightarrow \quad \dot{x}_1 = x_2$,

$$\dot{x}_2 = \frac{1}{m}(u - f) = \frac{1}{m}(u - k_1x_2 - v)$$

That is

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & -\frac{k_1}{m} \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u + \begin{pmatrix} 0 \\ -\frac{1}{m} \end{pmatrix} v$$

$$z = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

(b) Description of v:

$$\Phi_v(\omega) = |H(i\omega)|^2\Phi_e(\omega)$$

Thus $H(s) = \frac{\sqrt{k_0}}{s+|a|}$, i.e. $\dot{v}+|a|v = \sqrt{k_0}\, e$. Introduce an extra state $x_3 = v$ which results in a new state-space form with $\dot{x}_3 = -|a|x_3 + e$:

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -\frac{k_1}{m} & -\frac{1}{m} \\ 0 & 0 & -|a| \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{1}{m} \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 0 \\ \sqrt{k_0} \end{pmatrix} e$$

$$z = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x$$

The input-output form is

$$(p^2 + \frac{k_1 p}{m})\, z = \frac{1}{m} \left(u - \frac{\sqrt{k_0}}{p + |a|}\, e \right)$$

Go back

5.4

(a) With $\{A, B, C, N\}$ according to exercise 5.3 we get

$$\begin{aligned} \dot{x} &= Ax + Bu + Ne \\ y &= Cx + n \end{aligned}$$

where n has spectral density $\Phi_n \equiv 0.1$.

(b) A noise signal with the desired spectral density can be generated by a system with transfer function $G_n(s) = \frac{s}{s+|b|}$. The input is white noise with spectral density $\Phi_{w_n} = 0.1$. On state-space form we get

$$\begin{aligned} \dot{x}_4 &= -|b|x_4 + |b|w_n \\ n &= -x_4 + w_n \end{aligned}$$

The extended state-space form is

$$\begin{aligned} \dot{x} &= \begin{pmatrix} A & 0 \\ 0 & -|b| \end{pmatrix} x + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} N & 0 \\ 0 & |b| \end{pmatrix} \begin{pmatrix} e \\ w_n \end{pmatrix} \\ y &= \begin{pmatrix} C & -1 \end{pmatrix} x + w_n \end{aligned}$$

(c) Following the same procedure as in (b) we get a transfer function $G_n(s) = \frac{1}{s+|b|}$. The input is white noise with spectral density $\Phi_{w_n} = 0.1$. On state-space form we get

$$\dot{x}_4 + |b|x_4 = w_n.$$

The extended state-space form is

$$\begin{aligned} \dot{x} &= \begin{pmatrix} A & 0 \\ 0 & -|b| \end{pmatrix} x + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e \\ w_n \end{pmatrix} \\ y &= \begin{pmatrix} C & 1 \end{pmatrix} x \end{aligned}$$

Go back

5.5

(1) A model for w: A stepwise change results from $w = \frac{1}{s}v$ where v is a number of impulses.
Introduce the state x_w, $\dot{x}_w = v$.

(2) A model for n: Use a second order system with a resonance peak at $\omega_0 = 2\pi \cdot 2 = 4\pi$ rad/s and damping $\xi = 0.01$

$$n = \frac{\omega_0^2}{p^2 + 2\xi\omega_0 p + \omega_0^2} e$$

Introduce the states $x_{n1} = n$ and $x_{n2} = \dot{n}$

$$\dot{x}_n = \begin{pmatrix} \dot{x}_{n1} \\ \dot{x}_{n2} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\xi\omega_0 \end{pmatrix}}_{A_n} x_n + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B_n} e$$

We get the extended model

$$\dot{x}_u = \begin{pmatrix} \dot{x} \\ \dot{x}_w \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} A & N & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_n \end{pmatrix} x_u + \begin{pmatrix} B \\ 0 \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & B_n \end{pmatrix} \begin{pmatrix} v \\ e \end{pmatrix}$$

Go back

5.6

(a) Choose the states x_1 = acceleration and x_2 = speed. This results in the state-space form

$$\dot{x} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e$$

$$y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

(b)

$$\dot{\hat{x}} = \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \right] \hat{x} + \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} y$$

The matrix K is determined from the algebraic Riccati equation.

Go back

5.7

We get the state-space form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{=A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{=N} v_1$$

where $x_1 = x$, $x_2 = \dot{x}$.

The Kalman filter:

$$\dot{\hat{x}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \hat{x} + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} (y - C\hat{x})$$

where $C = (1 \;\; 0)$ (Case I) or $C = (0 \;\; 1)$ (Case II). The noise intensity is

$$\begin{aligned} R &= \begin{bmatrix} R_1 & R_{12} \\ R_{21} & R_2 \end{bmatrix} \\ &= \begin{cases} E\left[\begin{bmatrix} v(t) \\ e_1(t) \end{bmatrix} \begin{bmatrix} v(t) & e_1(t) \end{bmatrix}\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{in case I, and} \\ \\ E\left[\begin{bmatrix} v(t) \\ e_2(t) \end{bmatrix} \begin{bmatrix} v(t) & e_2(t) \end{bmatrix}\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{in case II.} \end{cases} \end{aligned}$$

The Kalman gain is

$$K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = PC^T R_2^{-1},$$

where $P = E[\tilde{x}(t)\tilde{x}^T(t)]$ is given by the ARE

$$AP + PA^T - PC^T R_2^{-1} CP + NR_1N^T = 0.$$

<u>Case I</u>:

$$P = \begin{pmatrix} 0.910 & 0.414 \\ 0.414 & 1.287 \end{pmatrix}$$

Case II:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The position x_1 is estimated more accurately in case I and the speed x_2 is estimated more accurately in case II.

Go back

5.8

Introduce the states

$$x(t) = \begin{pmatrix} \Theta(t) \\ \dot{\Theta}(t) \end{pmatrix}$$

and denote $\alpha = B/J$, $H = k/J$ and $\gamma = 1/J$. A state space model of the system is

$$\dot{x}(t) = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix}}_{A} x(t) + \underbrace{\begin{pmatrix} 0 \\ H \end{pmatrix}}_{B} \mu(t) + \underbrace{\begin{pmatrix} 0 \\ \gamma \end{pmatrix}}_{N} \tau_d(t)$$

$$y(t) = \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{C} x(t) + e_m(t)$$

The Riccati equation used used to compute the Kalman gain K is

$$AP + PA^T + NR_1N^T - (PC^T + NR_{12})R_2^{-1}(PC^T + NR_{12}) = 0.$$

Using $R_{12} = 0$ (process- and measurement noise independent), A, N and C as above, $R_1 = v_d$ and $R_2 = v_m$ gives

$$\begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} P + P \begin{pmatrix} 0 & 0 \\ 1 & -\alpha \end{pmatrix} + \begin{pmatrix} 0 \\ \gamma \end{pmatrix} v_d \begin{pmatrix} 0 & \gamma \end{pmatrix} - P \begin{pmatrix} 1 \\ 0 \end{pmatrix} v_m^{-1} \begin{pmatrix} 1 & 0 \end{pmatrix} P.$$

With

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix},$$

the components of this matrix equation become

$$\begin{aligned} 2p_{12} - \frac{p_{11}^2}{v_m} &= 0 \\ p_{22} - \alpha p_{12} - \frac{p_{11}p_{12}}{v_m} &= 0 \\ -2\alpha p_{22} + \gamma^2 v_d - \frac{p_{12}^2}{v_m} &= 0 \end{aligned}$$

If we eliminate p_{12} and p_{22} we get

$$\frac{p_{11}^4}{4v_m^3} + \frac{\alpha p_{11}^3}{v_m^2} + \frac{\alpha^2 p_{11}^2}{v_m} - \gamma^2 v_d = 0$$

Now introduce

$$p_{11} = v_m \cdot p'_{11}$$

which yields

$$\begin{aligned} {p'}_{11}^4 + 4\alpha {p'}_{11}^3 + 4\alpha^2 {p'}_{11}^2 - 4\gamma^2 \frac{v_d}{v_m} &= 0 \\ ({p'}_{11}^2 + 2\alpha p'_{11})^2 - 4\gamma^2 \frac{v_d}{v_m} &= 0 \\ {p'}_{11}^2 + 2\alpha p'_{11} - 2\gamma \sqrt{\frac{v_d}{v_m}} &= 0 \end{aligned}$$

Define $\beta = \gamma\sqrt{\frac{v_d}{v_m}}$ This results in

$$p'_{11} = -\alpha + \sqrt{\alpha^2 + 2\beta}$$

The solution is

$$P = v_m \begin{pmatrix} -\alpha + \sqrt{\alpha^2 + 2\beta} & \alpha^2 + \beta - \alpha\sqrt{\alpha^2 + 2\beta} \\ \alpha^2 + \beta - \alpha\sqrt{\alpha^2 + 2\beta} & -\alpha^3 - 2\alpha\beta + (\alpha^2 + \beta)\sqrt{\alpha^2 + 2\beta} \end{pmatrix}$$

The steady state Kalman gain $K = PC^T R_2^{-1}$ becomes

$$K = \begin{pmatrix} -\alpha + \sqrt{\alpha^2 + 2\beta} \\ \alpha^2 + \beta - \alpha\sqrt{\alpha^2 + 2\beta} \end{pmatrix}$$

and using the numerical values given we get

$$K = \begin{pmatrix} 40.36 \\ 814.3 \end{pmatrix}$$

The covariance matrix for the estimation error is

$$P = \begin{pmatrix} 40.36 \cdot 10^{-7} & 814.3 \cdot 10^{-7} \\ 814.3 \cdot 10^{-7} & 366.1 \cdot 10^{-5} \end{pmatrix}$$

Hence the filter for estimating Θ is

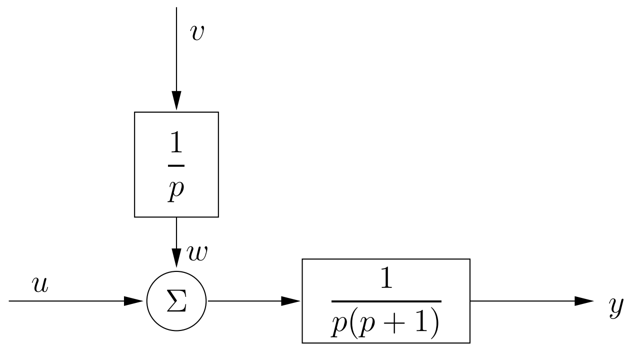
$$\dot{\hat{x}} = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} \hat{x} + \begin{pmatrix} 0 \\ H \end{pmatrix} \mu(t) + K \left(y - \begin{pmatrix} 1 & 0 \end{pmatrix} \hat{x} \right)$$

with K as above.

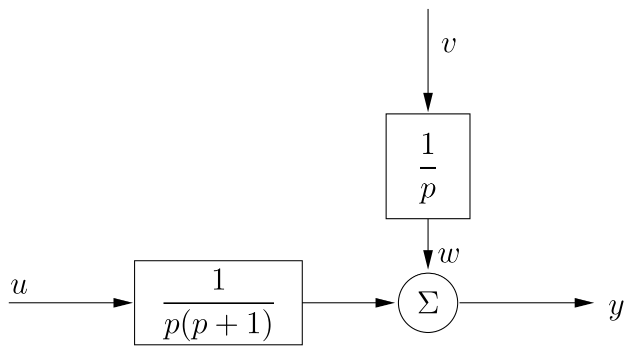
Go back

5.9

(i)



(ii)



$v(t)$ unit disturbance

(a) (i)

$$\dot{x} = \overbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}}^{A} x + \overbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}^{B} u + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v$$

$$y = \underbrace{\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}}_{C} x.$$

(ii)

$$\dot{x} = \overbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}^{A} x + \overbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}^{B} u + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v$$

$$y = \underbrace{\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}}_{C} x.$$

(b) (i) Offset in the motor voltage, step disturbance in the load

(ii) Measurement disturbance – error in the sensor for angular displacement

(c) (i)

$$S = \begin{pmatrix} B & AB & A^2B \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{not full rank}$$

(ii)

$$S = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{not full rank}$$

In (i) we can make x_3 unobservable by chosing $u = -Lx$ with $\ell_3 = 1$. This is not possible in (ii).

Go back

5.10

(a) The spectrum of the wind has low pass characteristics with bandwidth α. When α increases $v(t)$ behaves more and more like white noise, i.e. the gustiness increases. This can also be seen by studying the covariance function

$$R_v(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_v(\omega) e^{i\omega\tau} d\omega = e^{-\alpha|\tau|}, \quad \alpha > 0.$$

The covariance function gets more narrow when α increases, i.e. the correlation with neighboring values of $v(t)$ decreases and the gustiness increases.

(b) Using spectral factorization, the influence from the wind can be described as white noise $e(t)$ with intensity 1 filtered through a linear system with transfer function

$$H(s) = \frac{\sqrt{2/\alpha}}{1 + s/\alpha}$$

. We get $y = G(s)\,H(s)e$ where

$$G(s)\,H(s) = \frac{K\sqrt{2\alpha}}{(\alpha + s)(s^2 + s + 1)} = \frac{K\sqrt{2\alpha}}{s^3 + (1+\alpha)s^2 + (1+\alpha)s + \alpha}.$$

The variance of the output signal is

$$\begin{aligned}
\mathrm{Var}(y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(i\omega)H(i\omega)|^2 d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left| \frac{K\sqrt{2\alpha}}{(i\omega)^3 + (1+\alpha)(i\omega)^2 + (1+\alpha)i\omega + \alpha} \right|^2 d\omega \\
&= \frac{K^2(1+\alpha)}{1+\alpha+\alpha^2}.
\end{aligned}$$

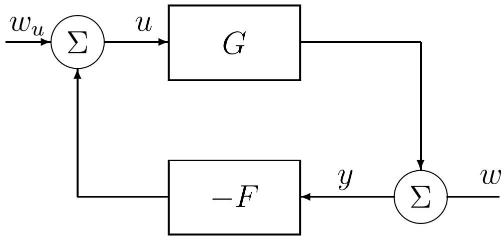
Thus the requirement can be formulated as $\frac{K^2(1+\alpha)}{1+\alpha+\alpha^2} > 1$.

Go back

6 The Closed-Loop System

6.1

Consider the block diagram



We have the relationships

$$y = (I + GF)^{-1}(w + Gw_u) = G_{wy}w + G_{w_u y}w_u$$

and

$$u = (I + FG)^{-1}(w_u - Fw) = G_{w_u u}w_u + G_{wu}w$$

which results in the input-output model

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} G_{w_u u} & G_{wu} \\ G_{w_u y} & G_{wy} \end{bmatrix} \begin{bmatrix} w_u \\ w \end{bmatrix}.$$

We also have

$$w_u = u + Fy$$

and

$$w = y - Gu$$

which results in the transfer function matrix

$$\begin{bmatrix} w_u \\ w \end{bmatrix} = \begin{bmatrix} I & F \\ -G & I \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}$$

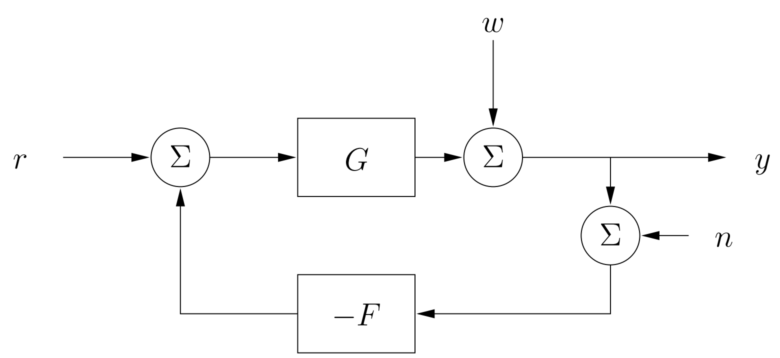
Thus, we have shown that

$$\begin{bmatrix} G_{w_u u} & G_{wu} \\ G_{w_u y} & G_{wy} \end{bmatrix}^{-1} = \begin{bmatrix} I & F \\ -G & I \end{bmatrix}$$

Alternative solution: Show that the matrix product is the identity matrix.

Go back

6.2



With the transfer functions

$$G = \frac{s-1}{s+2}, \quad F = \frac{s+2}{s-1}$$

we get

$$Y = G(R - F(Y+N)) + W \;\Rightarrow\; (1+GF)Y = GR - GFN + W$$

$$\Rightarrow\; Y = (1+GF)^{-1}GR - (1+GF)^{-1}GFN + (1+GF)^{-1}W$$

The closed-loop system, the sensitivity function and the comlementary sensitivity function are

$$G_c = G_{ry} = (1+GF)^{-1}G = \frac{s-1}{2s+3}$$
$$S = G_{wy} = (1+GF)^{-1} = \frac{s+1}{2s+3}$$
$$T = 1 - S = \frac{s+2}{2s+3}$$

and are all stable.

Internal stability?
Check the following transfer functions

$$
\begin{aligned}
G_{w_u u} &= (1+FG)^{-1} = \frac{s+1}{2s+3} \\
G_{wu} &= -(1+FG)^{-1}F = -\frac{(s+2)(s+1)}{(s-1)(2s+3)} \\
G_{w_u y} &= (1+GF)^{-1}G = \frac{s-1}{2s+3} \\
G_{wy} &= (1+GF)^{-1} = \frac{s+1}{2s+3}
\end{aligned}
$$

The systemet is not internally stable as G_{wu} is unstable.

Go back

7 Limitations in Control Design

7.1

(a) The complementary sensitivity function is given by

$$T(s) = \frac{F(s)G(s)}{1 + F(s)G(s)}$$

To compute the controller $F(s)$ which results in the desired $T(s)$ we express $F(s)$ as a function of $T(s)$ and $G(s)$ as follows. The above expression yields

$$F(s) = G^{-1}(s)\frac{T(s)}{1 - T(s)}$$

With the given $T(s)$ and $G(s)$ this results in

$$F(s) = \frac{5(s+1)}{s(s-3)}$$

The zero located in $s = 3$ will be cancelled. However, the transfer function from reference signal to control signal

$$U(s) = \frac{5(s+1)}{(s-3)(s+5)}R(s)$$

has a pole located in $s = 3$ and is unstable.

(b) We can get a bandwidth of 5 rad/s if we keep the right-half plane zero and add a pole in $s = -3$, i.e.

$$T(s) = \frac{5}{s+5} \cdot \frac{3-s}{3+s}$$

In this case, the relationship

$$F(s) = G^{-1}(s)\frac{T(s)}{1 - T(s)}$$

yields

$$F(s) = -\frac{5(s+1)}{s(s+13)}$$

No pole-zer cancellation occurs and all closed-loop system transfer functions are stable.

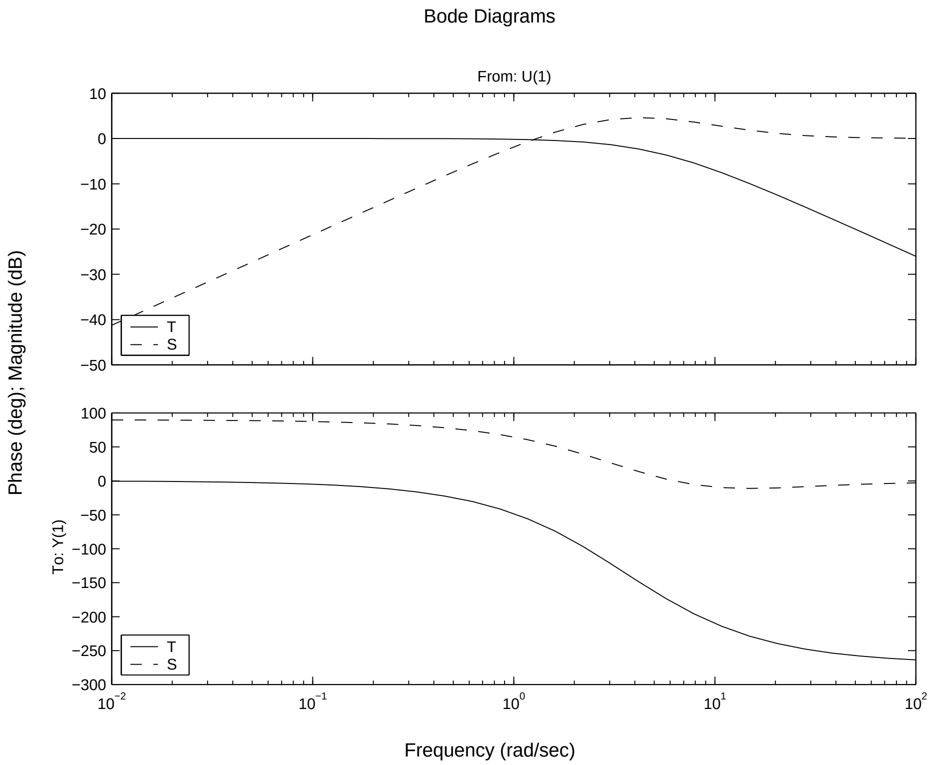
(c) With $F_r = F_y = F$ we get

$$G_c(s) = T(s) = \frac{5}{s+5} \cdot \frac{3-s}{3+s}$$

The sensitivity function is

$$S(s) = 1 - T(s) = \frac{s(s+13)}{(s+3)(s+5)}$$

The Bode diagrams for those transfer functions are shown in the figure below.



The bandwidth for G_c (i.e. T) is 5 rad/s as desired. However, the phase is -90° already at 2 rad/s. This means that the output signal will not follow the reference signal for frequencies above this. In addition, $|S(i\omega)| > 1$ for $\omega \geq 1.3$ rad/s, i.e. system disturbances are *amplified* for those frequencies.

Go back

7.2

According to the rules of thumb presented in the textbook the bandwidth of the closed-loop system cannot be greater than (1) half the magnitude of the right-half plane zero, in this case $3/2 = 1.5$ rad/s, (2) the inverse of the time-delay, here $1/1 = 1$ rad/s.

As a comparison we can study the crossover frequency. The crossover frequency is often close to the bandwidth. The transfer function for a system with a zero in $s = 3$ and a time-delay of 1 second can be expressed as

$$G(s) = e^{-s}(3-s)\bar{G}(s)$$

or

$$G(s) = e^{-s}\frac{(3-s)}{(3+s)}(3+s)\bar{G}(s)$$

The argument of the frequency response is

$$\arg G(i\omega) = -\omega - 2\arctan\frac{\omega}{3} + \arg((3+i\omega)\bar{G}(i\omega))$$

According to the assumptions the magnitude curve decreases monotonically and according to Bode's relation we get

$$\arg((3+i\omega)\bar{G}(i\omega)) \leq 0$$

This implies that

$$\arg G(i\omega) \leq -\omega - 2\arctan\frac{\omega}{3}$$

and the phase margin is

$$\varphi_m = \pi + \arg G(i\omega_c) \leq \pi - \omega_c - 2\arctan\frac{\omega_c}{3}$$

Let us study the case $\varphi_m = 0$ under the assumption that equality holds in the above inequality. Then

$$0 = \pi - \omega_c - 2\arctan\frac{\omega_c}{3}$$

i.e.

$$\omega_c \approx 2$$

Hence, the crossover frequency cannot be greater than 2 rad/s.

Go back

7.3

Assume that we have a zero close to the origin and a pole far from the origin in, both in the right-half plane. For example, we could have ($\epsilon << 1$)

$$G(s) = \frac{-\epsilon + s}{-\frac{1}{\epsilon} + s}$$

According to Theorem 7.4 in the textbook, the magnitude of the sensitivity function must have a peak in a neighborhood of ϵ. In addition, Theorem 7.6 says that the magnitude of the complementary sensitivity function must have a peak in the neighborhood of $1/\epsilon$.

Go back

7.4

(a) The requirements on $|S(i\omega)| = \bar{\sigma}(S(i\omega))$ and $|T(i\omega)| = \bar{\sigma}(T(i\omega))$ can be formulated as

$$|S(i\omega)| \leq \tfrac{1}{10}, \quad \omega \leq 0.1, \qquad\qquad |T(i\omega)| \leq \tfrac{1}{10}, \quad \omega \geq 2$$
$$|S(0)| \leq \tfrac{1}{100}$$

(b) The corresponding requirements on the loop gain GF_y is

$$|G(0)F_y(0)| > 100$$
$$|G(i\omega)F_y(i\omega)| > 10, \quad \omega \leq 0.1$$
$$|G(i\omega)F_y(i\omega)| < \frac{1}{10}, \quad \omega \geq 2$$

(c) The requirements in (a) can be reformulated using weighting functions W_S and W_T such that

$$|S(i\omega)| \leq |W_S^{-1}(i\omega)|, \quad \forall\omega$$
$$|T(i\omega)| \leq |W_T^{-1}(i\omega)|, \quad \forall\omega$$

If W_S^{-1} and W_T^{-1} are first order transfer functions

$$W_S^{-1}(s) = a_1\left(1 + \frac{s}{b_1}\right), \qquad W_T^{-1}(s) = \frac{a_2}{s}\left(1 + \frac{s}{b_2}\right)$$

we get, for example,

$$W_S^{-1}(s) = \frac{1}{100}(1 + 30\sqrt{11}s), \qquad W_T^{-1}(s) = \frac{\sqrt{2}}{10s}\left(1 + \frac{s}{2}\right)$$

(d) The minimal slope of the magnitude of the loop gain in the interval [0.1, 2] is approximately given by the line tangent to the forbidden regions in (b).

$$\text{Slope in the Bode plot:} \quad \frac{\log 0.1 - \log 10}{\log 2 - \log 0.1} \approx -1.53$$

This implies

$$\frac{\log 1 - \log 10}{\log \omega_c - \log 0.1} = -1.53 \quad \Rightarrow \quad \omega_c = 0.45 \text{ rad/s}$$

From Bode's relation we get

$$\arg GF_y \overset{<}{\approx} -1.53 \cdot \frac{\pi}{2} = -138^\circ$$

which results in a phase margin of approximately 40°.

A lower bound on $\|T\|_\infty$?

$$G(i\omega_c)F_y(i\omega_c) = 1 \cdot e^{-i \cdot 138^\circ} = -0.743 + 0.669i$$

$$|T(i\omega_c)| = \left|\frac{G(i\omega_c)F_y(i\omega_c)}{1 + G(i\omega_c)F_y(i\omega_c)}\right| \approx 1.4$$

$$\|T\|_\infty = \sup_\omega |T(i\omega)| \quad \Rightarrow \quad \|T\|_\infty \geq |T(i\omega)|, \quad \forall\omega$$

$$\Rightarrow \quad \|T\|_\infty \geq 1.4$$

(e)

$$|T(i\omega_c)| = 1.4$$

$$\left|W_T^{-1}(i\omega_c)\right| = \frac{0.14}{0.45}\sqrt{1 + \frac{0.45^2}{2^2}} = 0.32$$

It is impossible to find a feasible solution using this choice of weighting functions. Try weighting functions of higher order.

Go back

7.5

If the surface A_2 is greater than the surface A_1 we have that $\int_0^\infty \log|S(i\omega)|d\omega > 0$. According to Theorem 7.3, the loop gain $G(s)F_y(s)$ has unstable poles.

Go back

7.6

The first requirement implies that

$$|S(i\omega)| < 10^{-3} \qquad \omega \leq 2$$

where

$$S(s) = \frac{1}{1 + F(s)G(s)}$$

When $|F(i\omega)G(i\omega)|$ is large we approximately have

$$|S(i\omega)| \approx \frac{1}{|F(i\omega)G(i\omega)|}$$

which results in

$$|F(i\omega)G(i\omega)| > 10^3 \qquad \omega \leq 2$$

Furthermore, the system should be stable in spite of the model uncertainty

$$|\Delta G(i\omega)| \leq 100|G(i\omega)| \qquad \omega \geq 20$$

where $\Delta G(s)$ is the absolute model error in $G(s)$. Thus, the relative model error fulfills the inequality

$$\left|\frac{\Delta G(i\omega)}{G(i\omega)}\right| \leq 100$$

To preserve stability we must have

$$|T(i\omega)| < 10^{-2} \qquad \omega \geq 20$$

where

$$T(s) = \frac{F(s)G(s)}{1 + F(s)G(s)}$$

When $|F(i\omega)G(i\omega)|$ is small we approximately have

$$|T(i\omega)| \approx |F(i\omega)G(i\omega)|$$

and this results in

$$|F(i\omega)G(i\omega)| < 10^{-2} \qquad \omega \geq 20$$

To fulfill this requirement, the loop gain must decrease from 10^3 to 10^{-2} between the frequencies $\omega = 2$ to $\omega = 20$, i.e. 100 dB in a decade (slope -5). According to Bode's relation we have $\arg G(i\omega) \approx -5 \cdot 90°$ in this interval. This results in an unstable closed-loop system. We can not fulfill the requirements.

Go back

7.7

Go back

8 Controller Structures and Control Design

8.1

(a)

$$\text{RGA}(G(0)) = G(0) \,.*\, G^{-T}(0) = \begin{pmatrix} -\frac{5}{7} & \frac{12}{7} \\ \frac{12}{7} & -\frac{5}{7} \end{pmatrix}$$

(b) Avoid the pairs $u_1 \leftrightarrow y_1$ and $u_2 \leftrightarrow y_2$.

Go back

8.2

$$\text{RGA}(G(s)) = \begin{pmatrix} \frac{3}{s+4} & \frac{s+1}{s+4} \\ \frac{s+1}{s+4} & \frac{3}{s+4} \end{pmatrix}$$

For the frequency zero we get

$$\text{RGA}(G(0)) = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}$$

As all elements in the RGA$(G(0))$ are positive all combinations are possible. At the crossover frequency we get

$$\text{RGA}(G(10i)) = \begin{pmatrix} \frac{12-30i}{116} & \frac{104+30i}{116} \\ \frac{104+30i}{116} & \frac{12-30i}{116} \end{pmatrix}.$$

We have elements close to 1 if u_1 controls y_2 and u_2 controls y_1.

Go back

8.3

(a)
$$\text{RGA}(G(s)) = \begin{pmatrix} \frac{s-1}{s+1} & \frac{2}{s+1} \\ \frac{2}{s+1} & \frac{s-1}{s+1} \end{pmatrix}$$

yields

$$\text{RGA}(G(0)) = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}.$$

As we want to avoid pairing corresponding to negative elements in the RGA(0) we have to choose $u_1 \leftrightarrow y_2$ and $u_2 \leftrightarrow y_1$.

(b) As

$$G(0) = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$$

we choose $W_1 = G^{-1}(0)$ and $W_2 = I$. A controller that decouples the system in steady state is

$$F(s) = W_1 F^{\text{diag}}(s) W_2 = \begin{pmatrix} -F_{11}(s) & 2F_{22}(s) \\ -F_{11}(s) & F_{22}(s) \end{pmatrix}.$$

Go back

8.4

Ideally, we want $Q(s)$ to be $Q(s) = G^{-1}(s)$. A realizable choice is:

$$Q(s) = \frac{\tau s + 1}{K(\lambda s + 1)}.$$

This results in

$$F_y(s) = \frac{Q(s)}{1 - Q(s)G(s)} = \frac{\tau}{K\lambda}\left(1 + \frac{1}{\tau s}\right).$$

This is a PI controller with gain $K_{PI} = \frac{\tau}{K\lambda}$ and integration time $T_I = \tau$. The sensitivity function is

$$S(s) = 1 - G(s)Q(s) = \frac{\lambda s}{\lambda s + 1}$$

and the complementary sensitivity function is

$$T(s) = G(s)Q(s) = \frac{1}{\lambda s + 1}.$$

Thus, we have

$$|S(i\omega)| \leq 1 \quad \forall \omega$$

which seems to disagree with Bode's integral theorem. However, as the loop gain GF_y decreases as $1/\omega$ Bode's integral theorem is not applicable.

Go back

8.5

The system is a nonminimum phase system. We chose to replace the right-half plane zero with a zero mirrored in the imaginary axis.

$$Q(s) = \frac{s^2 + 5s + 6}{(6 + 3s)(\lambda s + 1)}.$$

This results in the controller

$$F(s) = \frac{s^2 + 5s + 6}{s(3\lambda s + 6(\lambda + 1))},$$

which can be rewritten as

$$F(s) = \frac{5}{6(1+\lambda)} \left(1 + \frac{6}{5s} + \frac{s}{5}\right) \frac{1}{\frac{3\lambda}{6(\lambda+1)}s + 1}.$$

This is a PID controller with a filter added to make it realizable.

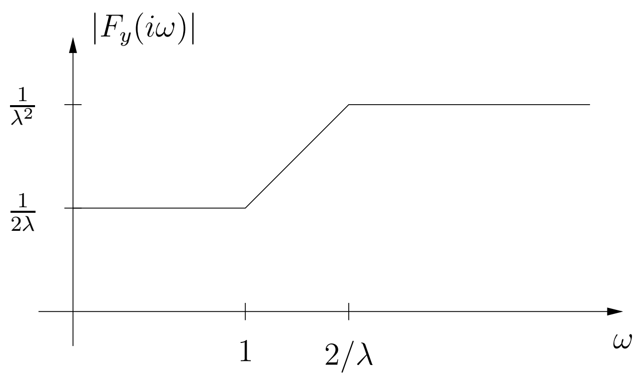
Go back

8.6

$$Q(p) = G^{-1}(p)\frac{1}{(\lambda p + 1)^2} = \frac{p(p+1)}{(\lambda p + 1)^2}$$

$$F_y(p) = (1 - Q(p)G(p))^{-1}Q(p) = \frac{p+1}{\lambda^2 p + 2\lambda} = \frac{1}{2\lambda} \cdot \frac{1+p}{1 + \frac{p}{2/\lambda}}$$

$$\Rightarrow \quad u = -\frac{1}{2\lambda} \cdot \frac{1+p}{1+\frac{p}{2/\lambda}} y$$



High bandwidth $\Rightarrow$ λ small $\Rightarrow$ $F_y(p) \approx \dfrac{1+p}{2\lambda}$ $\Rightarrow$ PD controller

Go back

8.7

(a) We have the transfer function matrix

$$G(s) = \frac{1}{s/20+1}\begin{pmatrix} \frac{9}{s+1} & 2 \\ 6 & 4 \end{pmatrix}.$$

The poles are the least common denominator of the minors

$$\begin{array}{ll} g_{11}(s) = \frac{9}{(s/20+1)(s+1)} & g_{12}(s) = \frac{2}{s/20+1} \\ g_{21}(s) = \frac{6}{s/20+1} & g_{22}(s) = \frac{4}{s/20+1} \end{array}$$

and the minor

$$\det G(s) = \frac{1}{(s/20+1)^2}\left(\frac{36}{s+1} - 12\right) = \frac{24(1-s/2)}{(s/20+1)^2(s+1)}.$$

This results in the poles -20, -20 and -1. The zeros are given by $\det G(s)$ normalized with the pole polynomial. This yields a zero located at $s = 2$. We have to take proper care of the right-half plane zero in the IMC design.

(b)

$$\begin{aligned} G^{-1}(s) &= (s/20+1)\begin{pmatrix} \frac{9}{s+1} & 2 \\ 6 & 4 \end{pmatrix}^{-1} \\ &= \frac{(s/20+1)(s+1)}{24(-s/2+1)}\begin{pmatrix} 4 & -2 \\ -6 & \frac{9}{s+1} \end{pmatrix} \end{aligned}$$

Mirror the right-half plane zero of $G(s)$ in the imaginary axis and add the factor $(\lambda s+1)$.

$$Q(s) = \frac{(s/20+1)(s+1)}{24(\lambda s+1)(s/2+1)}\begin{pmatrix} 4 & -2 \\ -6 & \frac{9}{s+1} \end{pmatrix}$$

This results in the controller

$$F_y(s) = (I - Q(s)G(s))^{-1}Q(s) = \frac{(s/20+1)(s+1)}{24s(\lambda s/2+\lambda+1)}\begin{pmatrix} 4 & -2 \\ -6 & \frac{9}{s+1} \end{pmatrix}$$

Go back

8.8

The inverse of the system is:

$$G^{-1}(s) = \frac{1}{-s+1}\begin{pmatrix} (s+1)(s+2) & -3(s+1)^2 \\ -(s+1)(s+2) & 2(s+1)(s+2) \end{pmatrix}$$

Mirror the right-half plane zero in the imaginary axis and form $Q(s)$

$$Q(s) = \frac{1}{\lambda s+1}\frac{-s+1}{s+1}G^{-1}(s) = \frac{1}{\lambda s+1}\begin{pmatrix} s+2 & -3(s+1) \\ -(s+2) & 2(s+2) \end{pmatrix}$$

The controller is given by $F_y = (I - QG)^{-1}Q$. The corresponding sensitivity function is

$$S = I - GQ = I - \frac{1}{\lambda s+1}\frac{-s+1}{s+1}I = \begin{pmatrix} \frac{s(\lambda s+2+\lambda)}{(s+1)(\lambda s+1)} & 0 \\ 0 & \frac{s(\lambda s+2+\lambda)}{(s+1)(\lambda s+1)} \end{pmatrix}$$

$|S(i\omega)| \to 0$ dï¿½ $\omega \to 0 \quad \Rightarrow \quad$ The controller has integral action.

Go back

8.9

(a) Since system is quadratic, i.e the number of inputs equals the number of outputs, the zeros can be determined as the poles of $G^{-1}(s)$. This gives

$$G^{-1}(s) = \frac{1}{\det G(s)}\begin{pmatrix} \dfrac{1}{s+1} & \dfrac{-3}{s+2} \\ \dfrac{-\alpha}{s+1} & \dfrac{2}{s+1} \end{pmatrix} =$$

$$= \frac{1}{s(2-3\alpha)+4-3\alpha}\begin{pmatrix} (s+1)(s+2) & -3(s+1)^2 \\ -\alpha(s+1)(s+2) & 2(s+1)(s+2) \end{pmatrix}$$

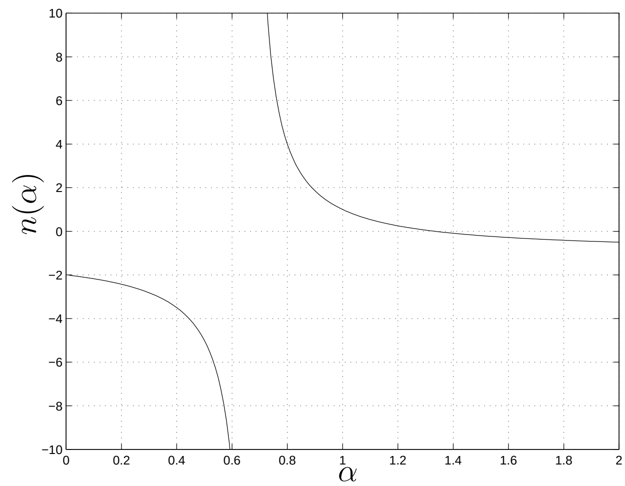
The pole polynomial becomes

$$s(2-3\alpha)+4-3\alpha$$

and hence the zero polynomial is given by

$$n(\alpha) = \frac{3\alpha - 4}{2 - 3\alpha}.$$

The location of the zero is shown in the figure below.



Figur 7: Location of the zero as function of α.

(b) In order to obtain that $F(s)G(s)$ is diagonal, $F(s)$ has to contain the inverse of $G(s)$. For $2/3 < \alpha < 4/3$ we have $n(\alpha) > 0$, i.e. the zero is located in the right half plane. This means that $F(s)$ is unstable, which should be avoided if possible.

(c) Using static decoupling F is a constant matrix containing $G(0)^{-1}$. For $\alpha = 4/3$ the inverse does not exist, since the rows of $G(0)$ becomes linear dependent.

Go back

8.10

(a) The RGA can be computed as follows.

```
>> s = tf('s');
G = [1/(s+2) 2/(s+4); 1/(s+1) 1/(s+2)];
G0 = freqresp(G,0);
RGA = G0.*inv(G0).'

RGA =

    -1     2
     2    -1
```

Negative elements in the diagonal of the RGA indicate that it will not be possible the control system using a diagonal regulator.

(b) The transfer function matrix of the closed loop system is given by

$$G_c(s) = (I + G(s)F(s))^{-1}G(s)F(s)$$

and using Matlab it can be computed as

```
>> F = diag([5 5]);
Gc = feedback(G*F,eye(2));
pole(Gc)

ans =

 -14.4265
  -2.3060 + 1.3712i
  -2.3060 - 1.3712i
   0.0385
 -14.4265
  -2.3060 + 1.3712i
  -2.3060 - 1.3712i
   0.0385
```

which means that the poles of the closed loop system are -14.4, $-2.3 \pm 1.37i$ and 0.04. Since there is a pole in the right half plane the system is unstable.

Note: $G(s)$ has 4 poles and a constant regulator does not add any poles. The number of poles are doubled by the functions `feedback` and `tf`. The extra poles are removed by using state space representation and the command `minreal`.

```
>> pole(minreal(ss(Gc)))

ans =

 -14.4265
   0.0385
  -2.3060 + 1.3712i
  -2.3060 - 1.3712i
```

(c) The problem can be modified by renumbering the output signals, i.e. switch the columns in $G(s)$. This gives the modified transfer function matrix

$$\bar{G}(s) = \begin{pmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{1}{s+2} & \frac{2}{s+4} \end{pmatrix}$$

The RGA becomes

```
Gb = [1/(s+1) 1/(s+2); 1/(s+2) 2/(s+4)];
G0 = freqresp(Gb,0);
RGA = G0.*inv(G0).'

RGA =

     2    -1
    -1     2
```

and the closed loop system becomes

$$\bar{G}_c(s) = (I + \bar{G}(s)F(s))^{-1}\bar{G}(s)F(s)$$

and this gives the poles

```
>> Gc = feedback(Gb*F,eye(2));
pole(Gc)

ans =

 -16.2449
  -4.8756
  -1.4398 + 0.9522i
  -1.4398 - 0.9522i
 -16.2449
```

```
-4.8756
-1.4398 + 0.9522i
-1.4398 - 0.9522i
```

i.e. the closed loop is stable.

Go back

9 Minimization of Quadratic Criteria: LQG

9.1

(a) We have $A = B = C = N = M = 1$. Thus, the Riccati equation for the Kalman filter is

$$2P + R_1 - \frac{P^2}{R_2} = 0,$$

The positive semidefinite solution is $P = R_2 + R_2\sqrt{1 + \frac{R1}{R2}}$. Hence, the Kalman gain is

$$K = \frac{1}{R_2}P = 1 + \sqrt{1 + \frac{R_1}{R_2}} = 1 + \sqrt{1+\beta}.$$

Analogously we get for the state feedback

$$L = \frac{1}{Q_2}S = 1 + \sqrt{1 + \frac{Q_1}{Q_2}} = 1 + \sqrt{1+\alpha}.$$

This results in the controller

$$F_y(p) = L(p - 1 + L + K)^{-1}K = \frac{(1+\sqrt{1+\alpha})(1+\sqrt{1+\beta})}{p + 1 + \sqrt{1+\alpha} + \sqrt{1+\beta}}.$$

(b) The poles of the transfer functions of the closed loop system are given by the eigenvalues of $A - BL$ and $A - KC$ respectively. Thus, the poles are

$$-\sqrt{1+\alpha} \quad \text{and} \quad -\sqrt{1+\beta}.$$

A small penalty on u (α large) results in a pole far from the origin in the left-half plane. A large penalty on u results in a pole close to -1, i.e. the pole of the original system mirrored in the imaginary axis. The pole located in $-\sqrt{1+\beta}$ can be affected in a similar way using the Kalman filter design parameters.

Go back

9.2

(a) Introduce the state $x = z$. Then the system can be written as

$$\begin{aligned}\dot{x} &= -x + u + v,\\ y &= x + e,\\ z &= x,\end{aligned}$$

i.e. $A = -1$, $B = 1$, $M = 1$, $Q_1 = q_1$ and $Q_2 = 1$. $Q_{12} = 0$. We cannot measure the state but we mesure y. According to the separation theorem V is minimized if we

(i) Estimate the state $\hat{x}(t)$ using a Kalman filter.

(ii) Use feedback according to $u(t) = -L\hat{x}(t)$, where L is computed using LQ theory.

Thus,

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$$

where $K = PC^T R_2^{-1}$ and P is the positive semidefinite solution to

$$AP + PA^T + NR_1N^T - PC^TR_2^{-1}CP = 0.$$

In our case this is a scalar equation

$$P^2 + 2P - r_1 = 0$$

with the solution

$$P = -1 + \sqrt{1 + r_1},$$

i.e.

$$K = -1 + \sqrt{1 + r_1}.$$

Use the feedback $u = -L\hat{x}$ with $L = Q_2^{-1}B^TS$, where S is the solution to

$$A^TS + SA + M^TQ_1M - SBQ_2^{-1}B^TS = 0$$

As $M = 1, Q_1 = q_1$ and $Q_2 = 1$ we get

$$L = S = -1 + \sqrt{1 + q_1}.$$

The loop gain is

$$\begin{aligned}G(s)F_y(s) &= \frac{1}{s+1}L\frac{1}{1+s+L+K}K =\\ &= \frac{(-1+\sqrt{1+r_1})(-1+\sqrt{1+q_1})}{(s+1)(s-1+\sqrt{1+r_1}+\sqrt{1+q_1}}\end{aligned}$$

(b) The parameters r_1 and q_1 influence the loop gain in the same way due to symmetry.

(c)

$$G(s)F_y(s) = \frac{(-1+\sqrt{1+r_1})(-1+\sqrt{1+q_1})}{(s+1)(s-1+\sqrt{1+r_1}+\sqrt{1+q_1})}$$

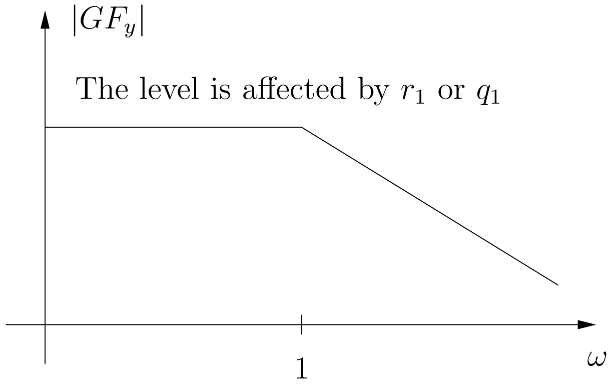
What happens when r_1 or $q_1 \to \infty$?

$$r_1 \to \infty \quad \Rightarrow \quad G(s)F_y(s) = \frac{(-1+\sqrt{1+q_1})}{(s+1)\frac{(s-1+\sqrt{1+r_1}+\sqrt{1+q_1})}{-1+\sqrt{1+r_1}}} \to \frac{-1+\sqrt{1+q_1}}{s+1}.$$

Analogously we get

$$\lim_{q_1 \to \infty} G(s)F_y(s) = \frac{-1+\sqrt{1+r_1}}{s+1}$$

By varying q_1 and/or r_1 we can shape the loop gain according to the sketch below:



Go back

9.3

State-space form:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$
$$z = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$
$$y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x$$

The weighting matrices are $Q_1 = 1$ and $Q_2 = \eta$. The Riccati equation is:

$$A^T S + SA + M^T Q_1 M - SBQ_2^{-1} B^T S = 0$$

Define

$$S = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix},$$

This results in

$$\begin{pmatrix} 0 & 0 \\ s_1 & s_2 \end{pmatrix} + \begin{pmatrix} 0 & s_1 \\ 0 & s_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \frac{1}{\eta} \cdot \begin{pmatrix} s_2^2 & s_2 s_3 \\ s_2 s_3 & s_3^2 \end{pmatrix} = 0$$

The positive definite solution is

$$\begin{aligned} s_1 &= \sqrt{2} \cdot \eta^{1/4} \\ s_2 &= \eta^{1/2} \\ s_3 &= \sqrt{2} \cdot \eta^{3/4} \end{aligned}$$

This yields the optimal feedback

$$\begin{aligned} L = Q_2^{-1} B^T S &= \frac{1}{\eta} \cdot \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2}\eta^{1/4} & \eta^{1/2} \\ \eta^{1/2} & \sqrt{2} \cdot \eta^{3/4} \end{pmatrix} \\ &= \frac{1}{\eta} \cdot (\eta^{1/2} \quad \sqrt{2}\eta^{3/4}) = (\eta^{-1/2} \quad \sqrt{2} \cdot \eta^{-1/4}) \end{aligned}$$

The poles are the eigenvalues of $A - BL$. Define $\mu = \eta^{-1/4} \quad \Rightarrow \quad L = \begin{pmatrix} \mu^2 & \sqrt{2} \cdot \mu \end{pmatrix}$. This results in

$$0 = \det \begin{pmatrix} s & -1 \\ \mu^2 & s + \sqrt{2} \cdot \mu \end{pmatrix} = s^2 + \sqrt{2}\mu s + \mu^2,$$

i.e.

$$\begin{aligned} s &= -\frac{\mu}{\sqrt{2}} \pm \sqrt{\frac{\mu^2}{2} - \mu^2} = -\frac{\mu}{\sqrt{2}} \pm i \cdot \frac{\mu}{\sqrt{2}} = \\ &= -\frac{\mu}{\sqrt{2}} \cdot (1 \pm i) = -\frac{1}{\sqrt{2} \cdot \eta^{1/4}} \cdot (1 \pm i) \end{aligned}$$

If η decreases the poles will be placed further away from the origin. This results in an increased input signal $u(t)$. Compare this result to the criterion.

Go back

9.4

Description of the system:

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ H \end{pmatrix} \mu(t) + \begin{pmatrix} 0 \\ \nu \end{pmatrix} \tau_d(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t) + v_m(t)$$

The steady state Riccati equation is

$$0 = S \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & -\alpha \end{pmatrix} S + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + S \begin{pmatrix} 0 \\ H \end{pmatrix} \frac{1}{\rho} \begin{pmatrix} 0 & H \end{pmatrix} S$$

Component by component we get

$$0 = 1 - \frac{H^2}{\rho} s_{12}^2$$
$$0 = -\frac{H^2}{\rho} s_{12} s_{22} + s_{11} - \alpha s_{12}$$
$$0 = -\frac{H^2}{\rho} s_{22}^2 + 2 s_{12} - 2\alpha s_{22}$$

The positive definite solution is

$$s_{11} = \frac{\sqrt{\rho}}{H} \sqrt{\alpha^2 + \frac{2H}{\sqrt{\rho}}}$$
$$s_{12} = \frac{\sqrt{\rho}}{H}$$
$$s_{22} = -\frac{\rho}{H^2} \left(\alpha - \sqrt{\alpha^2 + \frac{2H}{\sqrt{\rho}}} \right)$$

why the feedback gain L is given by

$$L = Q_2^{-1} B^T S = \frac{1}{\rho} \begin{pmatrix} 0 & H \end{pmatrix} \begin{pmatrix} \sqrt{\frac{\rho}{H}} \sqrt{\alpha^2 + \frac{2H}{\sqrt{\rho}}} & \frac{\sqrt{\rho}}{H} \\ \frac{\sqrt{\rho}}{H} & \frac{\rho}{H^2}(-\alpha + \sqrt{\alpha^2 + \frac{2H}{\sqrt{\rho}}}) \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{\sqrt{\rho}} & \frac{1}{H} \left(-\alpha + \sqrt{\alpha^2 + \frac{2H}{\sqrt{\rho}}} \right) \end{pmatrix}$$

The separation theorem states that it is optimal to use the estimated states in the feedback. Thus, the optimal feedback is

$$\mu(t) = -\begin{pmatrix} \frac{1}{\sqrt{\rho}} & \frac{1}{H} \left(-\alpha + \sqrt{\alpha^2 + \frac{2H}{\sqrt{\rho}}} \right) \end{pmatrix} \hat{x}$$

Go back

9.5

First find a state-space realization of the system

$$G(s) = \frac{1}{s(s+1)} \qquad \Leftrightarrow \qquad \ddot{y} + \dot{y} = u$$

Let $x_1 = y$, $x_2 = \dot{y}$ $\quad \Rightarrow$

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) \\ y(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x(t) + n(t) \end{aligned}$$

We want good robustness properties around the frequency $\omega = 0.5$ rad/s, i.e. we want the magnitude of the complementary sensitivity function $T(s)$ to be small at this frequency. As $T(s)$ is the transfer function from the measurement noise $n(t)$ to the output signal $y(t)$ we can proceed as follows:

If we estimate $\hat{x}(t)$ using the Kalman filter we will minimize the covariance matrix of the estimation error. The model we use for $n(t)$ will tell us for which frequencies the measurments of $y(t)$ are inaccurate. The Kalman filter will suppress measurements at those frequencies, i.e $|T(i\omega)|$ will be small.

As the Kalman gain K does not influence the closed-loop system $G_c(s)$, we can choose Q_1 and Q_2 to get a desired $G_c(s)$.

If we study the transfer function of the Kalman filter, i.e. the transfer function from $y(t)$ to $\hat{y}(t)$, we get an indication of how the mesurement noise affects the suppression.

We want the noise model to have much energy around the frequency $\omega = 0.5$ rad/s. One such model is $n(t) = H(p)w(t)$ where $w(t)$ is white noise, the poles of $H(s)$ are located at $s = -0.01 \pm 0.5i$ and there is a zero at $s = 0$, i.e.

$$H(s) = \frac{K_n s}{s^2 + 0.02s + 0.2501}.$$

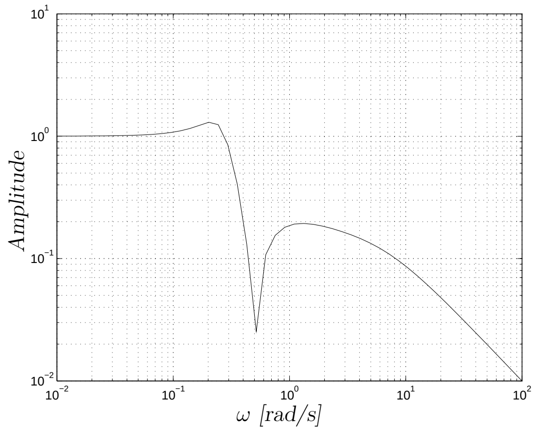
Using controllable canonical form we get

$$\begin{aligned}\dot{x}_n(t) &= \begin{pmatrix} -0.02 & -0.2501 \\ 1 & 0 \end{pmatrix} x_n(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} w(t) \\ n(t) &= \begin{pmatrix} K_n & 0 \end{pmatrix} x_n(t)\end{aligned}$$

Extending the original state-space form with the noise model yields

$$\begin{aligned}\dot{\bar{x}}(t) &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -0.02 & -0.2501 \\ 0 & 0 & 1 & 0 \end{pmatrix} \bar{x}(t) + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} w(t) \\ y(t) &= \begin{pmatrix} 1 & 0 & K_n & 0 \end{pmatrix} \bar{x}(t).\end{aligned}$$

If this model, with an appropriate value of K_n, is used to compute the Kalman gain K, the magnitude curve of the transfer function from $y(t)$ to $\hat{y}(t) = \hat{x}_1(t)$ will look as in the figure below. Signals at frequencies around $\omega = 0.5$ rad/s are heavily attenuated.



Go back

9.6

Let G be the system, F the controller, y the output signal and v the disturbance. This results in

$$y = \frac{1}{1+GF} v = Sv$$

$$u = \frac{F}{1+GF}v$$

where S is the sensitivity function.

F is chosen such as the criterion

$$J(F) = E\{y(t)^2 + \alpha u(t)^2\}, \quad \alpha > 0$$

is minimized given that $\Phi_v(\omega) = \delta(\omega)$.

The criterion can be written as

$$J(F) = \int_{-\infty}^{\infty} \Phi_y(\omega) + \alpha\Phi_u(\omega)\, d\omega = \frac{1}{(1+G_0F_0)^2} + \alpha\frac{F_0^2}{(1+G_0F_0)^2}.$$

where F_0 and G_0 are the stationary gains of the controller and the system respectively. As F minimizes $J(F)$ we have

$$\frac{\partial V}{\partial F} = 0 = \frac{2\alpha F_0 - 2G_0}{(1+G_0F_0)^3}$$

which yields $F_0 = G_0/\alpha$. Thus, the sensitivity function at $\omega = 0$ is

$$S_0 = \frac{1}{1+G_0F_0} = \frac{1}{1+G_0^2/\alpha}.$$

Go back

9.7

(a) How to solve the problem:

(i) Compute the Kalman filter: $\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$

(ii) Use feedback: $u = -L\hat{x}$, where L is computed using LQ theory.

(i) + (ii) results in the controller $F_y = L(sI - A + BL + KC - KDL)^{-1}K$.

(i) State-space description: Let $x_1 = z,\ x_2 = \nu,\ v_1 = v,\ v_2 = e$ and $x = (x_1, x_2)^T$. This results in

$$\dot{x} = \underbrace{\begin{pmatrix} -1 & 1 \\ 0 & -\epsilon \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{B} u + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{N} v_1$$
$$y = \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{C} x + v_2$$

Furthermore, we have

$$\begin{aligned} R_1 &= \Phi_{v_1}(\omega) = \Phi_v(\omega) = 1 \\ R_2 &= \Phi_{v_2}(\omega) = \Phi_e(\omega) = 1 \\ R_{12} &= \Phi_{v_1 v_2} = 0 \end{aligned}$$

The Kalman filter is given by: $K = PC^T R_2^{-1}$ with P according to

$$AP + PA^T + NR_1N^T - PC^T R_2^{-1} CP = 0$$

Define $P = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}$.

This results in $\lim_{\epsilon \to 0} P = \begin{pmatrix} \sqrt{3} - 1 & 1 \\ 1 & \sqrt{3} \end{pmatrix}$ and

$K = PC_T R_2^{-1} = \begin{pmatrix} \sqrt{3} - 1 \\ 1 \end{pmatrix}$

(ii) Compute L such that

$$\min_L \int_0^\infty x_1^2(t) + u^2(t)\, dt = \min_L \int_0^\infty y^T Q_1 y + u^T Q_2 u\, dt$$

where $Q_1 = Q_2 = 1$.

The optimal L is given by: $L = Q_2^{-1} B^T S$ where S is the positive semidefinite solution of

$$A^T S + SA + C^T Q_1 C - SBQ_2^{-1} B^T S = 0.$$

Define $S = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}$.

This results in $\lim_{\epsilon \to 0} S = \begin{pmatrix} \sqrt{2} - 1 & 1 - \frac{1}{\sqrt{2}} \\ 1 - \frac{1}{\sqrt{2}} & * \end{pmatrix}$ and

$L = Q_2^{-1} B^T S = \begin{pmatrix} \sqrt{2} - 1 & 1 - \frac{1}{\sqrt{2}} \end{pmatrix}$

The LQG controller is

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x}) \\ u = -L\hat{x} \end{cases}$$

with K and L as above.

The static gain of the sensitivity function ($\epsilon = 0$):
(i) and (ii) yields

$$F_y = L(sI - A + BL + KC - KDL)^{-1}K = \left\{ \begin{array}{c} D = 0 \\ s = 0 \end{array} \right\} = 1$$

$$S(0) = \frac{1}{1 + F_y(0)G(0)} = \frac{1}{1+1} = \frac{1}{2}.$$

(b) Compute L using LTR(y):

$$L_{ltr} = Q_2^{-1}B^TS$$
$$S : A^TS + SA + C^T\rho Q_2 C - SBQ_2^{-1}B^TS = 0$$

$$\Rightarrow S = \begin{pmatrix} \sqrt{1+\rho} - 1 & \frac{\sqrt{1+\rho}-1}{\sqrt{1+\rho}+\epsilon} \\ \frac{\sqrt{1+\rho}-1}{\sqrt{1+\rho}+\epsilon} & * \end{pmatrix} \Rightarrow L_{ltr} = \begin{pmatrix} \sqrt{1+\rho} - 1 & \frac{\sqrt{1+\rho}-1}{\sqrt{1+\rho}+\epsilon} \end{pmatrix}$$

The static gain of the sensitivity function:

$$S(0) \to \frac{\epsilon}{\sqrt{3}\epsilon + 1}$$

when $\rho \to \infty$. It is necessary to let $\epsilon \to 0$ to get $S(0) \to 0$, i.e. introducing an integrator into the system.

Go back

9.8

Define the matrices

$$\dot{x} = \overbrace{\begin{pmatrix} 0 & 1 & -1 \\ -\frac{1}{2}w_0^2 & -0.01 & 0.01 \\ \frac{1}{2}w_0^2 & 0.01 & -0.01 \end{pmatrix}}^{A} x + \overbrace{\begin{pmatrix} 0 \\ w_0 \\ 0 \end{pmatrix}}^{B} u$$

$$z = \underbrace{\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}}_{M} x,$$

where

$$w_0^2 = \frac{k}{50}.$$

The Bode plot for $k = 1$ is given. There is a resonance peak at $w_0 \approx 0.14$.

Introduce measurement noise: Let $y = z + v_2$, where v_2 is colored measurement noise. Using the feedback $u = -L\hat{x} + \tilde{p}r$ we can write z as

$$z = G_c r - T v_2 + \tilde{s} v_1.$$

The robustness criterion implies that $|T(i\omega_0)|$ should be small to handle large errors in k. A large spectrum for v_2 at $w = w_0$ will force T to be small at this frequency. Let v_2 be colored noise with a peak in the spectrum at w_0. This can be achieved by chosing poles in $-0.01 \pm 0.14i$ and a zero in 0, i.e.

$$v_2 = \frac{k_2 p}{p^2 + 0.021p + 0.02} w,$$

where w is white noise.

State-space representation of v_2:

$$\dot{x}_v = \underbrace{\begin{pmatrix} -0.02 & -0.02 \\ 1 & 0 \end{pmatrix}}_{A_v} x_v + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{B_v} w$$

$$v_2 = \underbrace{\begin{pmatrix} k_2 & 0 \end{pmatrix}}_{C_v} x_v$$

The extended model is

$$\dot{\bar{x}} = \begin{pmatrix} A & 0 \\ 0 & A_v \end{pmatrix} \bar{x} + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ B_v \end{pmatrix} w, \qquad \bar{x} = \begin{pmatrix} x \\ x_v \end{pmatrix}$$

$$y = \begin{pmatrix} M & C_v \end{pmatrix} \bar{x}$$

$$z = \begin{pmatrix} M & 0 \end{pmatrix} \bar{x}$$

$$v_2 = \begin{pmatrix} 0 & C_v \end{pmatrix} \bar{x}$$

with A, B, M, A_v, B_v, C_v as above.

Go back

9.9

The loop gain is

$$L(sI - A)^{-1}B = \frac{18}{(s-1)(s+2)}$$

The Nyquist curve will approach the origin with the angle −180°. An LQ controller always approaches the origin with −90°.

Go back

9.10

The system has the following controllability and observability matrices

$$\mathcal{C} = \begin{pmatrix} -4 & -12 \\ 8 & 24 \end{pmatrix}, \ \mathcal{O} = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix},$$

respectively. Thus, the system is neither controllable nor observable. Since $V(T)$ tends to the quadratic norm of the LQG problem as $T \to \infty$, we must have $V(\infty) = \infty$.

Go back

9.11

Go back

9.12

Go back

9.13

Go back

9.14

(a) According to Equation (9.7a) in the text book

$$L = Q_2^{-1} B^T S$$

where

$$0 = A^T S + SA + M^T Q_1 M - SBQ_2^{-1} B^T S$$

Here $A = \alpha$, $B = 1$ $M = 1$ $Q_1 = 1$ and $Q_2 = \rho$. This gives

$$0 = 2\alpha S + 1 - \frac{S^2}{\rho}$$

which implies

$$S^2 - 2\rho\alpha S - \rho = 0$$

with solution

$$S = \rho\alpha \overset{+}{(-)} \sqrt{(\alpha\rho)^2 + \rho}$$

This gives

$$L = \alpha + \sqrt{\alpha^2 + 1/\rho}$$

(b) Using the result from above gives for the case $\alpha = 1$

$$L = 1 + \sqrt{1 + 1/\rho}$$

i.e. $L \to 2$, while the case $\alpha = -1$ gives

$$L = -1 + \sqrt{1 + 1/\rho}$$

i.e. $L \to 0$. For $\alpha = 1$ the open loop system is unstable and has to be stabilized using feedback, and hence $u = 0$ will not work.

Go back

9.15

The choices (iii) and (iv) give the same gain vector L since $J_{(iii)} = 0.1\, J_{(iv)}$. The L that minimizes $J_{(iii)}$ will also minimize $J_{(iv)}$. The figures (A) and (C) show the same simulation results. The matrices in (i) put less weight on the input u which implies a faster settling, i.e. (B). The choice (ii) puts a weight on the velocity which implies a slower response, i.e. (D).

Answer: (i) – B, (ii) – D, (iii) – A and C, (iv) – A and C

Go back

9.16

(a) The system has the poles

$$0 \quad 0 \quad -0.0850 + 0.7435i \quad -0.0850 - 0.7435i$$

Since the system has two poles in the origin the system is not asymptotically stable.

(b) With

```
Q1 = diag([0 0 0 1]);
Q2 = diag(1);
L = lqr(A,B,Q1,Q2)
```

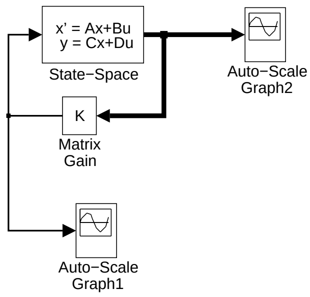
the feedback gain becomes

$$L = (-7.0973 \quad 2.0419 \quad 8.1531 \quad 1)$$

and the poles of the closed loop system (i.e. the eigenvalues of $A - BL$) becomes

$$\begin{array}{c} -0.1809 + 0.8271i \\ -0.1809 - 0.8271i \\ -0.4368 + 0.2915i \\ -0.4368 - 0.2915i \end{array}$$

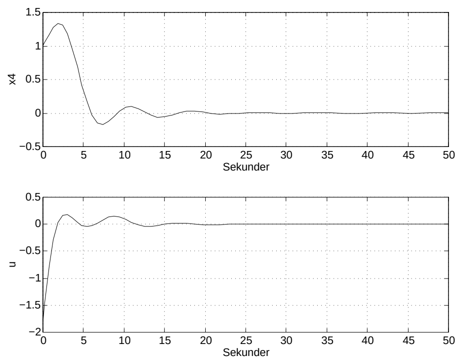
The closed loop system can be simulated using the model below (note, the LTI system is a system with the dynamics $\dot{x} = Ax + Bu$ with all states as output since they are required for the feedback, hence a system defined by the matrices (A,B,eye(4),zeros(4,1)). If one wants to look at a specific state, such as the controlled state, one would have to place a matrix gain outside the LTI block to create this signal. Alternatively, we



can create and simulate the system, and output the signals of interest, directly in MATLAB

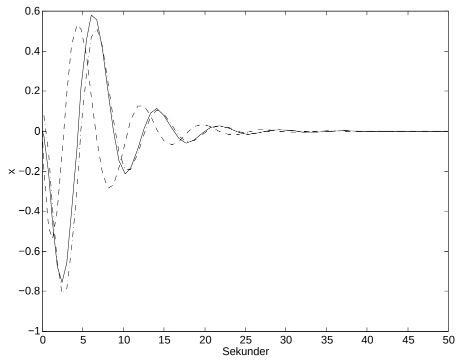
```
Gc_r_to_x = ss(A-B*L,B,eye(4),zeros(4,1))
[x,t] = initial(Gc_r_to_x,x0);
plot(t,x)
Gc_r_to_zu = ss(A-B*L,B,[0 0 0 1;-L],zeros(2,1))
[z_and_u,t] = initial(Gc_r_to_zu,x0);plot(t,z_and_u)
plot(t,z_and_u)
```

The controlled state x_4 and u are given by the figure and it can be seen



that all signals tend to zero. The other states are given in the figure below, where $x_1 \leftrightarrow$ solid line, $x_2 \leftrightarrow$ dashed line, $x_3 \leftrightarrow$ dash-dotted line

$$\begin{array}{lcl} x_1 & \leftrightarrow & \text{solid line} \\ x_2 & \leftrightarrow & \text{dashed line} \\ x_3 & \leftrightarrow & \text{dash-dotted line} \end{array}$$



(c) Increasing Q_2 reduces u and decreasing Q_2 increases u.

(d) An example of matrices that gives a feedback such that the conditions are fulfilled is given by

$$Q_1 = \begin{pmatrix} 250 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q_2 = 1$$

The feedback gain becomes

$$L = (-6.8105 \quad 7.5015 \quad 22.0860 \quad 1)$$

Go back

9.17

(a) The requirements are fulfilled by the state feedback

$$u(t) = -Lx(t)$$

where

$$L = \begin{pmatrix} -4.4721 & -4.9405 & 19.1028 & 6.1811 \end{pmatrix}$$

which is achieved from LQ-minimization using

$$Q_1 = \begin{pmatrix} 20 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad Q_2 = 1$$

The poles of the closed loop system, i.e the eigenvalues of $A - BL$, all have the absolute value 2.36.

(b) The closed loop system has the characteristic equation

$$\lambda^4 + l_4\lambda^3 + l_3\lambda^2 - 7l_2\lambda - 7l_1 = 0$$

The closed loop system is stable if all roots are located (strictly) in the left half of the complex plane. A necessary condition (but not sufficient) is that all coefficients in the polynomial are strictly positive. A loss of a measurement of state variable can be interpreted as $l_i = 0$ for some i, and this violates the condition.

Go back

10 Loop Shaping

10.1

The criterion to minimize is the $\mathcal{H}_2$ norm of G_{ec}. The system on state-space form:

$$\begin{aligned}\dot{x}_1 &= -x_1 + u \\ y &= x_1\end{aligned}$$

Weighting functions

$$W_u(s) = 5, \qquad W_T(s) = 0.5, \qquad W_S(s) = \frac{1}{s}$$

Form the extended system G_0:

$$\begin{aligned}z_1 &= W_u u = 5u \\ z_2 &= W_T G u = 0.5 x_1 \\ z_3 &= W_S(Gu + w) = x_2\end{aligned}$$

where x_2 is a new state defined as

$$x_2 = \frac{1}{p}(Gu + w) \quad \Leftrightarrow \quad \dot{x}_2 = x_1 + w$$

This yields

$$\begin{aligned}\dot{x} &= \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w \\ z &= \begin{pmatrix} 0 & 0 \\ 0.5 & 0 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x + w\end{aligned}$$

Check M and D

$$D^T \begin{pmatrix} M & D \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 5 \\ 0.5 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 25 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

Hence, define a new input signal $\tilde{u}$ as

$$\tilde{u} = (D^T D)^{1/2} u + (D^T D)^{-1/2} D^T M x = 5u \quad \Leftrightarrow \quad u = \frac{1}{5}\tilde{u}$$

This is just a scaling of the original input signal. Thus we get a new B matrix

$$\tilde{B} = \frac{1}{5} B$$

Solve the Riccati equation: $\quad A^T S + SA + M^T M - S\tilde{B}\tilde{B}^T S = 0$

Define

$$S = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}$$

which yields

$$\begin{pmatrix} -s_1 + s_2 & -s_2 + s_3 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -s_1 + s_2 & 0 \\ -s_2 + s_3 & 0 \end{pmatrix} + \begin{pmatrix} 0.25 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{25}\begin{pmatrix} s_1^2 & s_1 s_2 \\ s_1 s_2 & s_2^2 \end{pmatrix} = 0$$

Hence,

$$\begin{cases} -2s_1 + 2s_2 + 0.25 - \frac{1}{25}s_1^2 = 0 \\ -s_2 + s_3 - \frac{1}{25}s_1 s_2 = 0 \\ 1 - \frac{1}{25}s_2^2 = 0 \end{cases}$$

which has the positive semidefinite solution $s_1 = 4.686$, $s_2 = 5$ and $s_3 \geq \frac{5^2}{4.686}$. Thus, the state feedback for the scaled system is

$$\tilde{L} = \tilde{B}^T S = \begin{pmatrix} \frac{1}{5}s_1 & \frac{1}{5}s_2 \end{pmatrix} = \begin{pmatrix} 0.937 & 1 \end{pmatrix}$$

For the original system we get

$$L = \frac{1}{5}\tilde{L} = \begin{pmatrix} 0.187 & 0.2 \end{pmatrix}$$

The controller is

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + N(y - C\hat{x}) \\ u &= -L\hat{x} \end{aligned}$$

Go back

10.2

The criterion to minimize is the $\mathcal{H}_\infty$ norm of G_{ec}. The extended system is the same as in the previous exercise. The controller is $L = B^T S$, for the smallest value of γ for which

$$A^T S + SA + M^T M + S(\gamma^{-2} N N^T - BB^T)S = 0$$

has a positive semidefinite solution. If we solve this numerically using MATLAB we see that $\gamma \geq 5.12$ produce positive definite solutions. For $\gamma = 5.2$ we get

$$L = \begin{pmatrix} 2.6873 & 2.7632 \end{pmatrix}$$

10.3

(a) The frequency weights $W_S = \frac{1}{s}$ and $W_T = W_u = 1$ result in

$$\begin{aligned} z_1 &= W_u u = u \\ z_2 &= W_T G u = Cx \\ z_3 &= W_S(Gu + w) \qquad \Leftrightarrow \qquad \dot{z}_3 = Cx + w = y \end{aligned}$$

and

$$\begin{pmatrix} \dot{x} \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ z_3 \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w$$

$$y = \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} x \\ z_3 \end{pmatrix} + w$$

Controllers for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ cases can be computed using the expressions in the textbook.

(b) The observer is given by

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu \qquad \Leftrightarrow \qquad \hat{x} = (pI - A)^{-1} B u \\ \dot{\hat{z}}_3 &= C\hat{x} + (y - C\hat{x}) = y \qquad \Leftrightarrow \qquad \hat{z}_3 = \textstyle\int y \, d\tau \end{aligned}$$

The state feedback is

$$u = -\begin{pmatrix} L & -\alpha \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{z}_3 \end{pmatrix} = -L\hat{x} + \alpha \hat{z}_3 = -L(pI - A)^{-1} B u + \alpha \textstyle\int y \, d\tau$$

By solving for u we get the desired controller structure

$$u = \frac{\alpha}{1 + L(pI - A)^{-1} B} \int y \, d\tau$$

(c) If the system contains an integrator we have $\det(pI - A) = p \cdot \xi(p)$ which implies that

$$u = \frac{\alpha}{1 + \frac{1}{p\xi(p)} L(pI - A)^a B} \cdot \frac{1}{p} y = \frac{\alpha \xi(p)}{p\xi(p) + L(pI - A)^a B} y$$

i.e. the integral part of the controller is cancelled.

Go back

10.4

Go back

10.5

A normal requirement is to have a small sensitivity function for low frequencies, which means that C is excluded. Bode's integral theorem states that it is impossible to achieve that $| S(i\omega) | < 1 \quad \forall\, \omega$, which excludes B. Therefore A is the best alternative.

Go back

10.6

(a) Straightforward calculations give

$$S(s) = \frac{1}{1 + G(s)F(s)} = \frac{1}{1 + \frac{1}{s+1} K} = \frac{s+1}{s+1+K},$$

$$T(s) = \frac{G(s)F(s)}{1 + G(s)F(s)} = \frac{\frac{1}{s+1} K}{1 + \frac{1}{s+1} K} = \frac{K}{s+1+K} \quad \text{and}$$

$$G_{ru}(s) = \frac{F(s)}{1 + G(s)F(s)} = \frac{K}{1 + \frac{1}{s+1} K} = \frac{K(s+1)}{s+1+K}$$

(b)
- In Alternative II it is required that both S and T are small for low frequencies. Since $S + T = 1$ always holds this requirement is not realistic.
- In alternative III the requirement on T is the opposite to a correct specification for the complementary sensitivity function. Normally one requires, for robustness and measurement noise reasons, that T is small for high frequencies.
- Alternative I is realistically specified with small S for low frequencies and small T for high frequencies.

(c) The requirements can be fulfilled by e.g choosing $K = 9$.

Go back

10.7

Go back

12 Stability of Nonlinear Systems

12.1

The state variables $x_1 = y$ and $x_2 = \dot{y}$ yield

$$\dot{x} = \begin{pmatrix} x_2 \\ -0.2(1+x_2^2)x_2 - x_1 \end{pmatrix} = f(x),$$

and

$$\dot{V} = V_x f(x) = x_1\dot{x}_1 + x_2\dot{x}_2 = -0.2x_2^2(1+x_2^2).$$

Hence, $\dot{V} < 0$ except when $x_2 = 0$. If $x_2 \equiv 0$ we have that $x_1 = \text{constant} = 0$. Thus the zero solution is asymptotically stable.

Go back

12.2

No, $V(x) \geq 0,\ \forall x$ is not fulfilled.

Go back

12.3

The slopes

$$\begin{cases} k_1 = 0.5 \\ k_2 = 3 \end{cases} \Rightarrow$$

result in a circle going through the points $-1/3$ and -2.

Go back

12.4

The nonlinearity is bounded by two lines with slope $k_1 = 0$ and $k_2 = 1$ respectively. According to the circle criterion the closed loop system is stable if the Nyquist curve lies to the right of the line $s = -1$.

Go back

12.5

The circle criteria applies to a model with one nonlinear static block, and one linear dynamic block. Here, we have two linear blocks, so our first step is to convert it to the standard case.

To begin with, we note that we cannot simply move the constant K straight through the nonlinear block and put it next to the linear dynamics, since $Kf(u_1) \neq f(Ku_1)$. Nevertheless, we can move it to the linear dynamics block, but we have to go the other way around.

Let z denote the output from the linear dynamics and call the linear dynamics $H(s)$. We have $Z(s) = H(s)U_2(s)$, $u_1 = K(r - z)$, $u_2 = f(u_1)$. This can be written as $u_1 = Kr - Kz$, i.e., $u_1 = \tilde{r} - y$ if we define $\tilde{r} = Kr$ and $Y(s) = KZ(s) = (KH(s))U_2(s) = G(s)U_2(s)$. In this form, we have $u_2 = f(\tilde{r} - y)$, $Y(s) = G(s)U_2(s)$ which is exactly the type of loop analyzed using the circle criteria.

This was just a long-winded way of proving that you can pull K along the loop backwards and multiply it with the linear dynamics, remembering that when you pull it through the loop, you are redefining the external variable.

The nonlinearity is bounded by two lines with slope $k_1 = 0$ and $k_2 = \infty$ respectively. The linear dynamics must have a strictly positive real part. After simplifying the expression $G(i\omega) = \frac{K}{(i\omega)(i\omega+1)} = \frac{K(-i\omega)(-i\omega+1)}{(i\omega)(i\omega+1)(-i\omega)(-i\omega+1)}$ to find the real part, we arrive at

$$\text{Re } G(i\omega) = -\frac{K}{\omega^2 + 1} < 0, \quad \forall \omega$$

Notice that there is really no need to simplify as far as done here, as you know the denominator is a positive real number as it is an absolute value $zz^* = |z|^2$. All you have to do is to study the real part of the numerator after having multiplied with the conjugate of the denominator, since all you need is the sign of this term.

Hence, stability cannot be guaranteed for any $K > 0$.

Go back

12.6

The state variables $x_1 = \Phi, x_2 = \dot{\Phi}$ yield

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{mg}{J} \sin x_1 l \end{aligned}$$

The controller

$$l = l_0 + \varepsilon \Phi \dot{\Phi}$$

results in

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{mg}{J} \sin x_1 (l_0 + \varepsilon \Phi \dot{\Phi}) \end{aligned}$$

As a candidate Lyapunov function we use

$$V(x) = \frac{1}{2} J x_2^2 + mgl_0(1 - \cos x_1)$$

which corresponds to the energy of the system. We get

$$\dot{V} = Jx_2\dot{x}_2 + mgl_0 \sin x_1 \dot{x}_1 = -\varepsilon mg x_2^2 x_1 \sin x_1 \leq 0 \quad (-\pi/2 < x_1 \pi/2)$$

with $\dot{V} = 0$ only when $x_1 \equiv 0$ or $x_2 \equiv 0$. $x_1 \equiv 0 \Rightarrow x_2 = 0$ and $x_2 \equiv 0 \Rightarrow x_1 = 0$.

Answer: $\dot{V}(x) < 0$ for all$x_1 \neq 0, x_2 \neq 0$ implies that $x \rightarrow 0$.

Go back

12.7

The states $x_1 = y$, $x_2 = \dot{y}$ and the controller

$$u = -\operatorname{sgn}(ax_1 + bx_2)$$

yield

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2x_1 - 3x_2 - \operatorname{sgn}(ax_1 + bx_2)\end{aligned}$$

The Lyapunov function candidate

$$V(x) = (\frac{\alpha}{2}x_1^2 + \frac{\beta}{2}x_2^2)$$

results in

$$\dot{V} = (\alpha - 2\beta)x_1x_2 - 3\beta x_2^2 - \beta x_2 \operatorname{sgn}(ax_1 + bx_2)$$

Take, for example, $\alpha = 2, \beta = 1, a = 0, b = 1$, which result in

$$\dot{V} = -3x_2^2 - |x_2| \leq 0$$

Go back

12.8

The nonlinearity is

$$f(u) = u + \arctan(u)$$

The derivative of $f(u)$ is

$$f'(u) = 1 + \frac{1}{1 + u^2}$$

and has its maximum $f'(0) = 2$ for $u = 0$. $f'(u) \rightarrow 1$ dï¿½ $u \rightarrow \infty$. This implies

$$1 \leq \frac{u + \arctan(u)}{u} \leq 2$$

which means that the Nyquist curve of the linear part of the system must lie outside and not encircle the circle passing through -1 and $-1/2$.

Go back

12.9

According to the circle criterion the system is stable if the Nyquist curve lies outside and does not encircle the circle passing through$-4/3$ and $-4/7$.

According to the textbook the loop gain for an LQ controller lies outside and does not encircle the circle passing through 0 and -2. As this circle encompasses the above smaller circle the system is stable.

Go back

13 Phase Plane Analysis

13.1

a) From

$$\ddot{y} - (0.1 - \frac{10}{3}\dot{y}^2)\dot{y} + y + y^2 = 0$$

a reasonable guess is to try the states $x_1 = y$ and $x_2 = \dot{y}$. With these, we can write the system on state-space form

$$\begin{aligned}\dot{x}_1 &= x_2 = f_1(x_1, x_2)\\ \dot{x}_2 &= -x_1(1 + x_1) + x_2(0.1 - 10x_2^2/3) = f_2(x_1, x_2)\end{aligned}$$

b) To find the stationary points, we must find solutions to $\dot{x}(t)$. We call the stationary points $\bar{x}$ here.

$f(\bar{x}) = 0 \quad \Rightarrow \quad \bar{x}_2 = 0 \quad \text{and} \quad \bar{x}_1(1 + \bar{x}_1) = 0.$

$$\text{SP I}: \begin{cases} \bar{x}_1 = 0 \\ \bar{x}_2 = 0 \end{cases}, \qquad \text{SP II}: \begin{cases} \bar{x}_1 = -1 \\ \bar{x}_2 = 0 \end{cases}$$

c) To linearize the system around the stationary point we use

$$f(x) \approx f(\bar{x}) + f_x(\bar{x})(x - \bar{x}) = f_x(\bar{x})(x - \bar{x})$$

as $f(\bar{x}) = 0$. The matrix $f_x(x)$ is the Jacobian of$f$. The ij entry is $\frac{\partial f_i}{\partial x_j}(x)$:

$$\frac{\partial f_1}{\partial x_1} = 0, \qquad \frac{\partial f_1}{\partial x_2} = 1,$$

$$\frac{\partial f_2}{\partial x_1} = -1 - 2x_1, \qquad \frac{\partial f_2}{\partial x_2} = 0.1 - 10x_2^2.$$

Make the change of variables $z = x - \bar{x}$.

SP I:
Linear approximation $\dot{z} = Az$, with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0.1 \end{pmatrix}$$

The eigenvalues of the matrix A are given by

$$0 = \det(\lambda I - A) = \lambda(\lambda - 0.1) + 1,$$

i.e. we have unstable dynamics since we have postive real part on both eigenvalues.

$$\lambda = 0.05 \pm \sqrt{0.05^2 - 1}$$

SP II:
Linear approximation $\dot{z} = Bz$, where

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0.1 \end{pmatrix}$$

The eigenvalues of B,

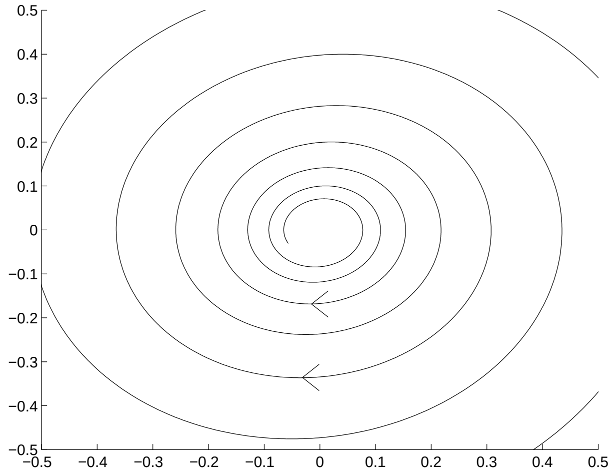
$$\begin{aligned} 0 &= \det(\lambda I - B) = \lambda(\lambda - 0.1) - 1 \\ \lambda &= 0.05 \pm \sqrt{0.05^2 + 1}, \quad \lambda_1 = -0.95, \quad \lambda_2 = 1.05 \end{aligned}$$

Once again unstable, since one of the eigenvalues is positive.

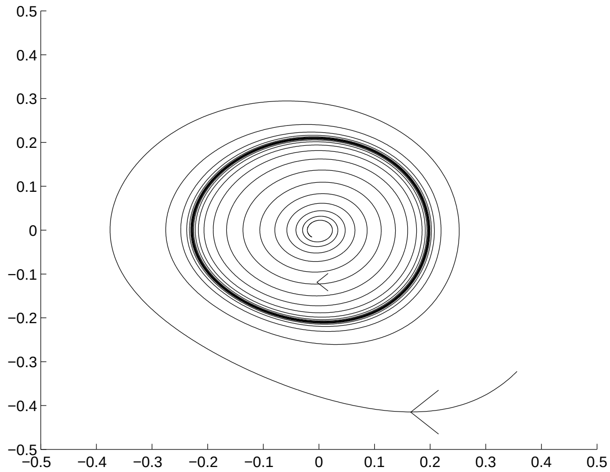
d) **SP I** :

The linear approximation has an unstable focus at (0,0). For unstable nodes the the nonlinear differential equation has a stationary of the same type as the linear approximation. Note that the linear approximation is only valid *close* to the stationary point.

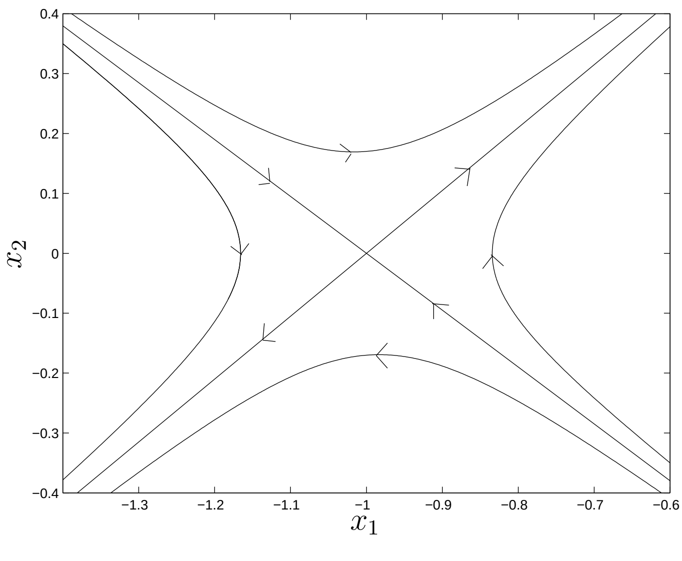
linear



nonlinear



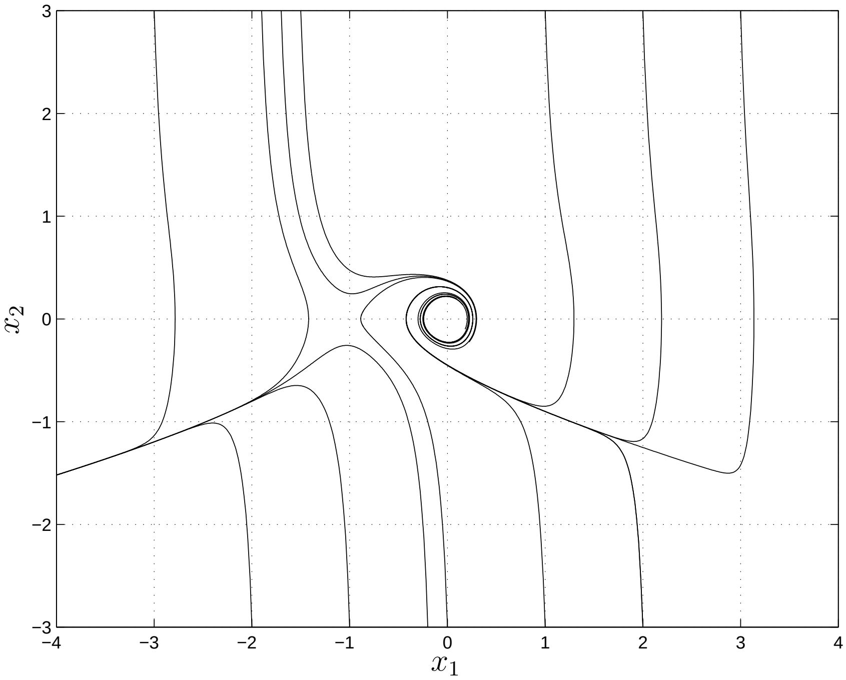
SP II:
The linearized approximation has a saddle point at $(-1, 0)$. This is true for the nonlinear differential equation too. The eigenvector corresponding to the stable eigenvalue is $(1, -0.95)$, and the eigenvalue corresponding to the unstable eigenvalue is $(1, 1.05)$.



Far away from the stationary points:
How do the trajectories behave far from the origin? Form the derivative,

$$\frac{dx_2}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{-x_1(1+x_1) + x_2(0.1 - 10x_2^2/3)}{x_2}$$

When x_1 is bounded and $x_2 \to \pm\infty$, we have that $\dot{x}_2/\dot{x}_1 \to \infty$. Hence, the trajectories ar vertical when $|x_2|$ grows and also when $x_2 \to 0$.



Go back

13.2

The nonlinearity is described by

$$f(x) = \begin{cases} x + a, & x < -a \\ 0, & -a \le x \le a \\ x - a, & x > a \end{cases}$$

The relationship between x and e is given by

$$p(p + B)x(t) = Ke(t)$$

i.e.

$$\ddot{x} + B\dot{x} = Ke$$

In addition we have that

$$e = u - f(x) = -f(x)$$

which yields

$$\ddot{x} + B\dot{x} + Kf(x) = 0$$

Introduce the states $x_1 = x$ and $x_2 = \dot{x}$. The state-space form is

$$\begin{aligned} \dot{x}_1 &= x_2 && (4) \\ \dot{x}_2 &= -Kf(x_1) - Bx_2 && (5) \end{aligned}$$

Partition into regions where $f(x)$ is linear.

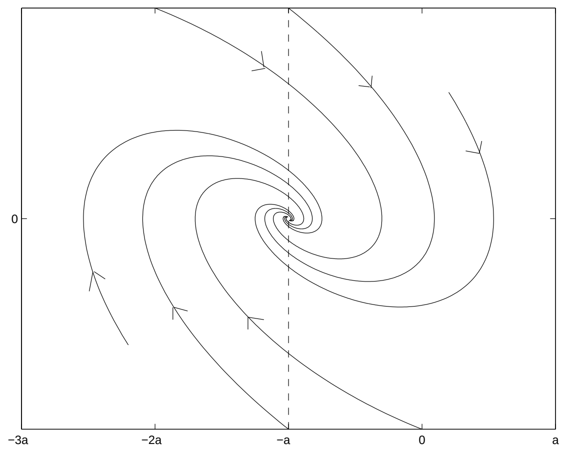
1. The region $x_1 < -a$:
   The stationary point is $x_1 = -a, x_2 = 0$. The cange of variables $z_1 = x_1 + a,\ z_2 = x_2$ results in linear state-space equations , $\dot{z} = Az$, where

   $$A = \begin{pmatrix} 0 & 1 \\ -K & -B \end{pmatrix}$$

   The eigenvalues of A are

   $$\lambda = -\frac{B}{2} \pm \sqrt{\frac{B^2}{4} - K}$$

   Thus, the point $x = (-a, 0)$ is a stable node or a stable focus. Sketching the phase portrait when $4K > B^2$, results in a stable focus.



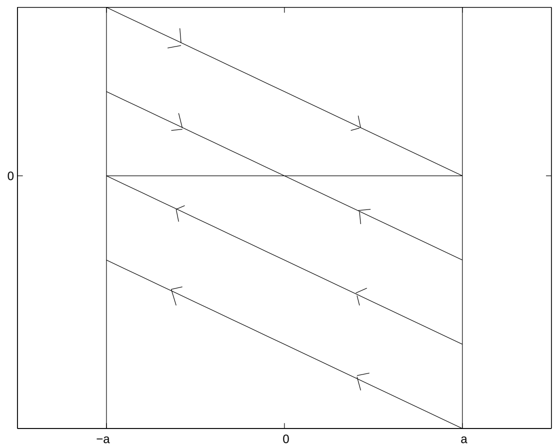
2. The region$-a \leq x_1 \leq a$:
   The stationary points are line segments : $-a \leq x_1 \leq a$ and $x_2 = 0$. The dynamic equations are

   $$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -Bx_2 \end{aligned}$$

Form the derivative

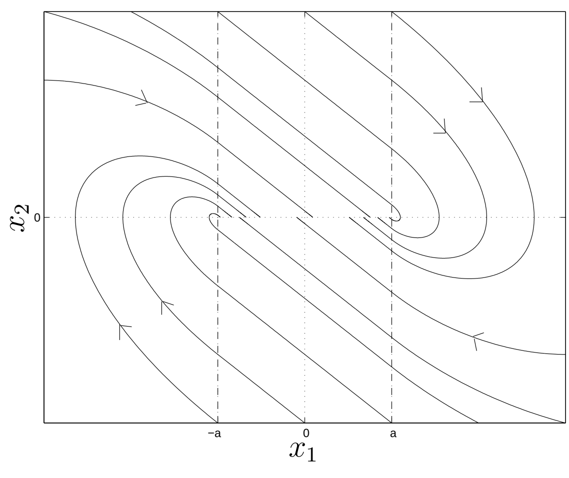
$$\frac{dx_2}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = -B$$

In the entire region the trajectories have the slope $-B$.



3. The region $x_1 > a$:
   stationary point $x_1 = a, x_2 = 0$. Make the change of variables $z_1 = x_1 - a, z_2 = x_2$. This results in the same state-space equation for z as for the case $x_1 < -a$, $\dot{z} = Az$. If $4K > B^2$, we have the unstable focus $x = (a, 0)$ analogously to the case $x_1 < -a$.

To get the phase portrait we need to join the three partial solutions found in 1, 2 and 3.



Go back

13.3

(a) Introduce $x_1 = y, x_2 = \dot{y}$, which yields

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\text{sgn } x_1\end{aligned}$$

There are no stationary points. The phase portrait is constructed in two steps depending on sgn x_1. When $x_1 > 0$ we have

$$\frac{dx_2}{dx_1} = -\frac{1}{x_2}$$

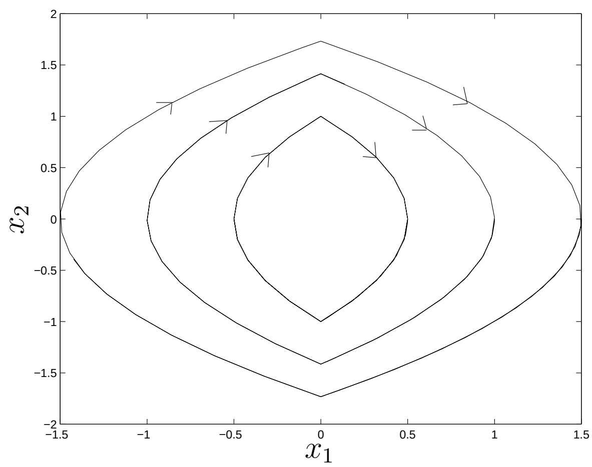
This separable differential equation has the solution

$$\frac{1}{2}x_2^2 + x_1 = \text{constant}$$

i.e. x_1 as a function of x_2 is a number of parabolas. When $x_1 < 0$ we get analogously

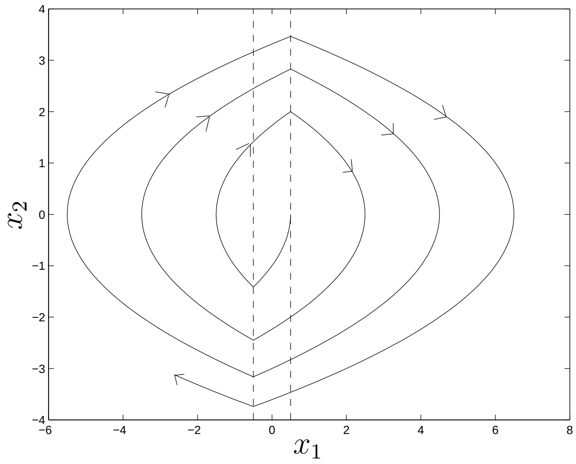
$$\frac{1}{2}x_2^2 - x_1 = \text{constant}$$

The phase portrait:



(b) For $x_1 > a$ we have $\frac{1}{2}x_2^2 + x_1 =$ konst and for $x_1 < -a$ we have $\frac{1}{2}x_2^2 - x_1 =$ konst. For the case $|x_1| \leq a$ the relay will have the same output as before it entered teh region, i.e. the parabola is continuing.

The phase portrait when $a = 0.5$:



Go back

13.4

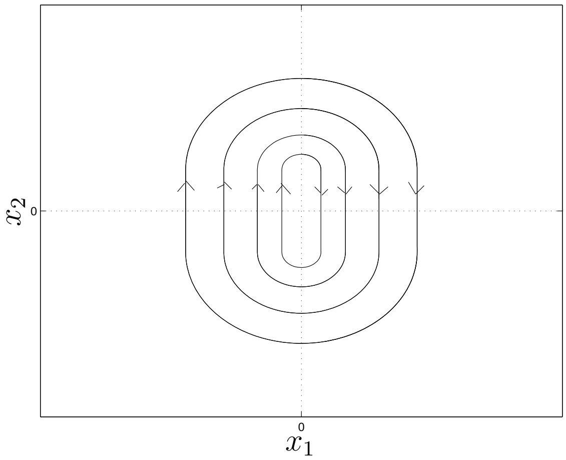
(a) Let $x_1 = y$ and let x_2 be the input to the nonlinearity. This results in the state-space form

$$\begin{aligned} \dot{x}_1 &= f(x_2) \\ \dot{x}_2 &= -x_1 \end{aligned}$$

where

$$f(x) = \begin{cases} x+1, & x < -1 \\ 0, & -1 \leq x \leq 1 \\ x-1, & x > 1 \end{cases}$$

We get centers in the stationary points $x = (0, -1)$ (for $x_2 \leq -1$) and $x = (0, 1)$ (for $x_2 \geq 1$). When $-1 < x_2 < 1$ we have $x_1 =$ constant. This results in the phase portrait

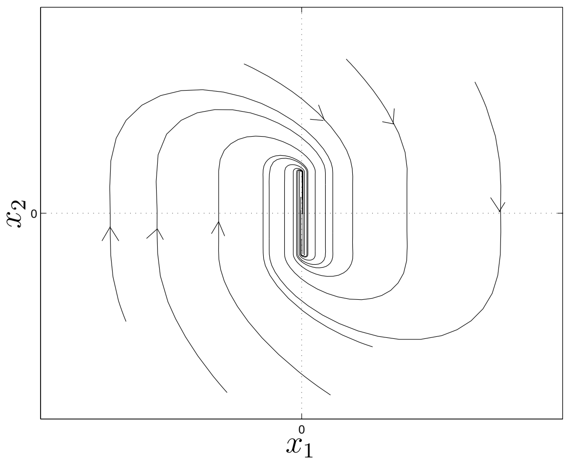


The car will not return to the desired position with proportional control.

(b) Now, we have the state-space form

$$\begin{aligned}\dot{x}_1 &= f(x_2)\\ \dot{x}_2 &= -x_1 - f(x_2)\end{aligned}$$

The difference from (a) is that the stationary points are stable focuses. Hence, we get the phase portrait



The car returns to the desired position.

Go back

13.5

$$\dot{x}_2 = 0 \quad \Rightarrow \quad 0 = -3x_2(1 + \frac{1}{6}x_1) + x_1 x_2 = \frac{1}{2}(x_1 - 6)x_2$$

i.e.

$$x_1 = 6 \quad \text{or} \quad x_2 = 0.$$

Two cases:

(i) $x_2 = 0$ and $\dot{x}_1 = 0 \quad \Rightarrow \quad 0 = 2x_1 - 0.2x_1^2 = 0.2(10 - x_1)x_1$,
i.e. $x_1 = 0$ or $x_1 = 10$.

(ii) $x_1 = 6$ and $\dot{x}_1 = 0 \quad \Rightarrow$

$$0 = 2 \cdot 6(1 + \frac{1}{6} \cdot 6) - 6 \cdot x_2 - 0.2 \cdot 6^2(1 + \frac{1}{6} \cdot 6) = 24 - 6x_2 - 14.4$$

Hence, the stationary points (SP) are

$$\text{SP I}: \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}, \quad \text{SP II}: \begin{cases} x_1 = 10 \\ x_2 = 0 \end{cases}, \quad \text{SP III}: \begin{cases} x_1 = 6 \\ x_2 = 1.6 \end{cases}$$

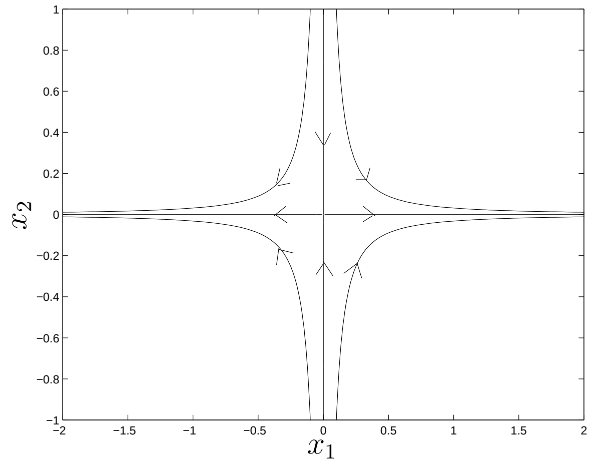
The Jacobian is

$$H(x) = f_x(x) = \begin{pmatrix} 2 - 0.4x_1 - x_2/(1 + x_1/6)^2 & -x_1/(1 + x_1/6) \\ x_2/(1 + x_1/6)^2 & -3 + x_1/(1 + x_1/6) \end{pmatrix}$$

SP I:
$x_1 = x_2 = 0$ yield

$$H_1 = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$$

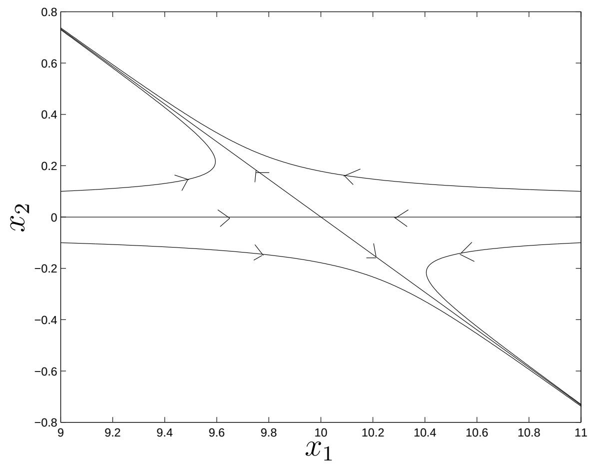
The origin is a saddle point with trajectories according to



SP II:
When $x_1 = 10$ and $x_2 = 0$ the Jacobian is

$$H_2 = \begin{pmatrix} -2 & -3.75 \\ 0 & 0.75 \end{pmatrix}$$

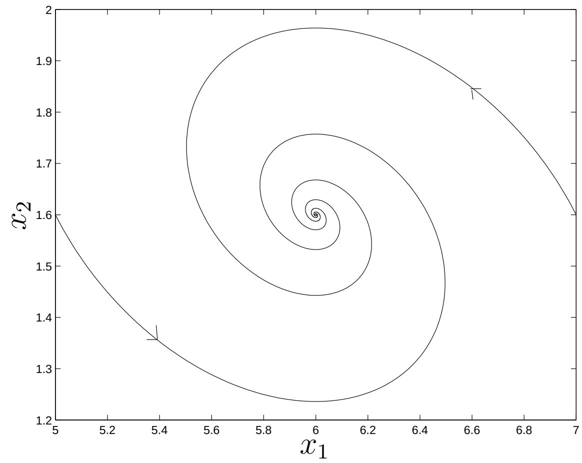
This is also a saddle point. The eigenvector corresponding to the unstable eigenvalue is $(3.75,\ -2.75)$, and the eigenvalue corresponding to the stable eigenvalue is $(1,\ 0)$. The phase portrait is



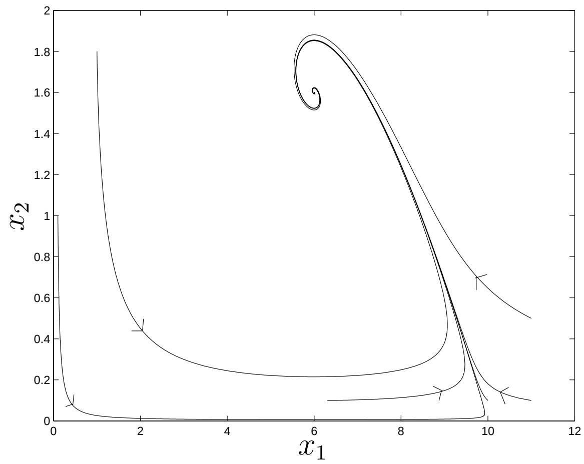
SP III:
$x_1 = 6,\ x_2 = 1.6$ yield

$$H_3 = \begin{pmatrix} -0.8 & -3 \\ 0.4 & 0 \end{pmatrix}$$

The eigenvalues are $-0.4 \pm 1.02i$. We have a stable focus with the phase portrait



IF we join these phase portraits together we get:



Go back

13.6

Introduce the states $x_1 = y$ and $x_2 = \dot{y} \ \Rightarrow$.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + f(x_2) \end{aligned}$$

For $x_2 > 0$ we have

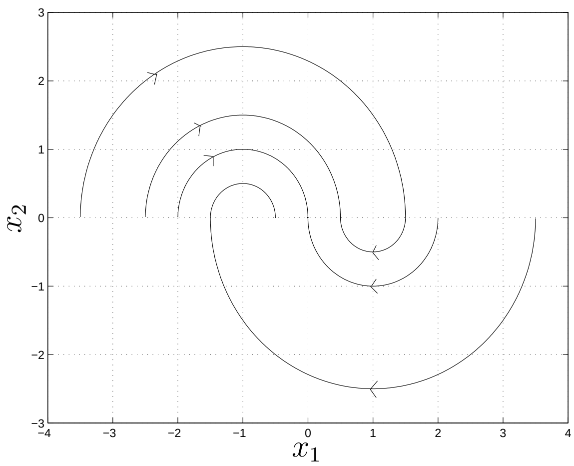
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - 1 \end{cases} \quad \Rightarrow \quad \text{stationary point} \quad (-1, 0)$$

The Jacobian is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

As the eigenvalues are $\pm i \ \Rightarrow \ x = (-1, 0)$ is a center.

For $x_2 < 0$ we get analogously that $x = (1, 0)$ is a center. If we join the phase portraits together we get

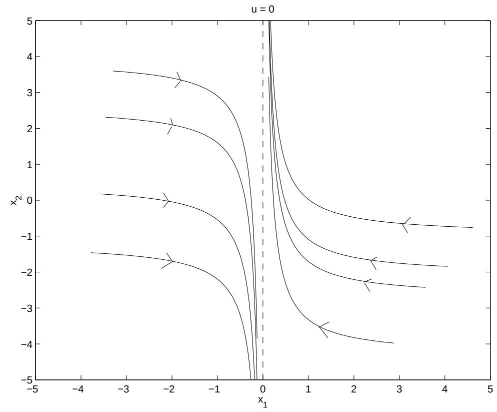


The system will tend to a point on the x_1-axis, i.e. $\dot{y}$ will tend to zero.

13.7

(a) We have stationary points for $x_1 = 0$, i.e. the entire x_2-axis, when $u = 0$. The trajectories are described by

$$\frac{dx_2}{dx_1} = -\frac{1}{x_1^2} \qquad \Leftrightarrow \qquad x_2 = \frac{1}{x_1} + C.$$



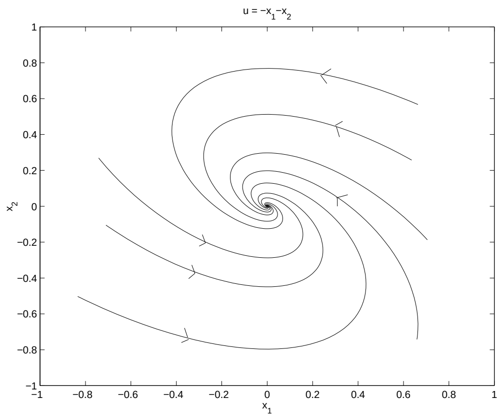
(b)

$$\dot{V} = V_x \dot{x} = -2x_1^4 + 2x_1 u + 2x_1 x_2 \quad \Rightarrow \quad \text{Vï¿½lj } u = -x_1 - x_2$$

This results in $\dot{V} = -2x_1^4 - 2x_1^2 < 0$ and the stationary point $x_1 = x_2 = 0$. The corresponding linearized system is

$$\dot{x} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} x, \quad \text{with the eigenvalues} \quad \lambda = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

The stationary point is a stable focus.



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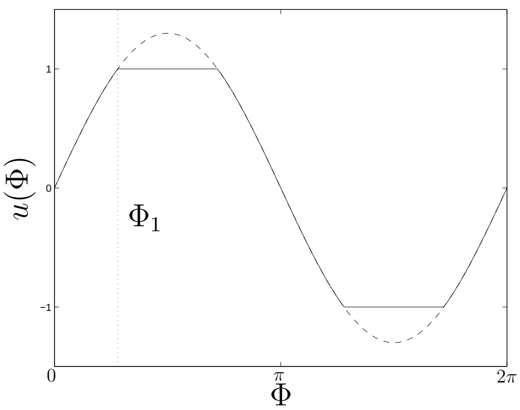
14 Oscillations and Describing Functions

14.1

The describing function of a saturation is given in the textbook, together with describing functions for some other nonlinearities. We will derive it here anyway to demonstrate how to do it.

1. Apply the signal $e(t) = C \sin \Phi$, where $\Phi = \omega t$, to the input of the saturation. If $C \leq 1$ the output signal after the saturation will be$u(t) = e(t)$. If $C > 1$ the output signal $u(t)$ will be the solid curve in the figure below:



   Here Φ_1 is given by $C \sin \Phi_1 = 1$, i.e. $\Phi_1 = \arcsin(1/C)$.

2. Compute the Fourier coefficients a_1 and b_1 as

$$a_1 = \frac{1}{\pi} \int_0^{2\pi} u(\Phi) \cos \Phi \, d\Phi \ , \quad b_1 = \frac{1}{\pi} \int_0^{2\pi} u(\Phi) \sin \Phi \, d\Phi$$

   As $u(\Phi)$ is an odd function and $\cos \Phi$ is even $a_1 = 0$. Utilizing symmetry we can write b_1 as

$$\begin{aligned} b_1 &= \frac{4}{\pi} \int_0^{\pi/2} u(\Phi) \sin \Phi \, d\Phi \\ &= \frac{4}{\pi} \left(\int_0^{\Phi_1} C \sin^2 \Phi \, d\Phi + \int_{\Phi_1}^{\pi/2} \sin \Phi \, d\Phi \right) \\ &= \frac{4C}{\pi} \left(\frac{\Phi_1}{2} - \frac{\sin 2\Phi_1}{4} + \frac{\cos \Phi_1}{C} \right) \end{aligned}$$

As $\sin 2\Phi_1 = 2\sin\Phi_1\cos\Phi_1$, $\sin\Phi_1 = 1/C$ and $\cos\Phi_1 = \sqrt{C^2-1}/C$ we get

$$b_1 = \frac{2C}{\pi}\left(\arcsin\frac{1}{C} + \frac{\sqrt{C^2-1}}{C^2}\right)$$

3. The describing function is given by $Y_f(C) = (b_1 + ia_1)/C$

$$Y_f(C) = \frac{2}{\pi}\left(\arcsin\frac{1}{C} + \frac{\sqrt{C^2-1}}{C^2}\right)$$

This is valid for $C > 1$, $Y_f(C) = 1$ when $C \leq 1$)

(a) The problem can be solved either by hand or by using Matlab and both alternatives will be presented here.

Alternative (i): According to the calculations above the describing function of the satuaration is real, starts in 1 for $C \leq 1$ and tends to zero when C tends to infinity. This means that $-1/Y_f$ will start in -1 and tend to $-\infty$ when C grows.

The transfer function of the linear part is

$$G(s) = \frac{10}{s(s+1)^2}$$

Since the system contains an intergrator the argument of $G(i\omega)$ will start at $-90°$ for low frequencies, and since the system has relative degree three the argument will tend to $-270°$. This implies that $G(i\omega)$ will cross the negative axis and there is a possibility that it will cross $-1/Y_f$. Using the fact that

$$\arg G(i\omega) = \arg 10 - \arg(i\omega(i\omega+1))^2 = 0 - 90° - 2\arctan\omega$$

we find that $\arg G(i\omega) = -180°$ (i.e. it crosses the negative real axis) for $\omega = 1$. Using also that

$$|G(i\omega)| = 10/(\omega(1+\omega^2))$$

we find that $\mid G(i1) \mid = 5$, i.e. $G(i\omega)$ crosses the negative real axis in the point -5, and there will hence be an intersection with $-1/Y_f$. In order to find the corresponding value of C we need to solve the equation

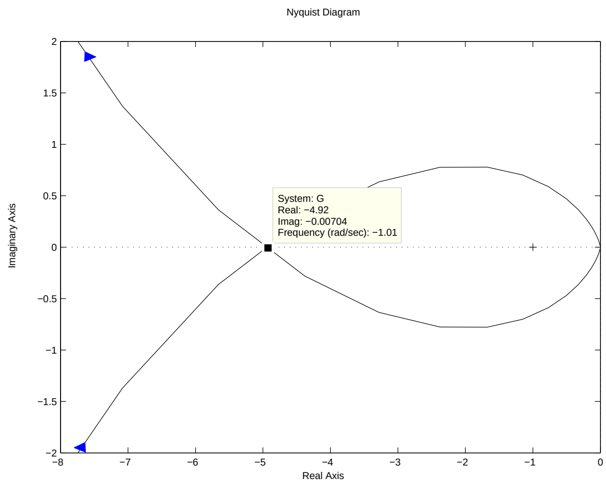
$$\frac{2}{\pi}\left(\arcsin\frac{1}{C} + \frac{\sqrt{C^2-1}}{C^2}\right) = 0.2$$

which has the approximate solution $C = 6.3$. For oscillations with $C < 6.3$ the curve $G(i\omega)$ will encircle $-1/Y_f$ and hence the amplitude of the oscillations will grow. Correspondingly, for oscillations with $C > 6.3$ the curve $G(i\omega)$ will not encircle $-1/Y_f$ and hence the amplitude of the oscillations will decay. Hence the describing function method predicts that there will be a limit cycle with angular frequency $\omega = 1$ and amplitude $C = 6.3$.

Alternative (ii): The Nyquist curve can be plotted in Matlab using

```
>> s=tf('s');
>> G=10/(s*(s+1)^2);
>> nyquist(G)
>> axis([-8 0 -2 2])
```

and by clicking in the plot one gets

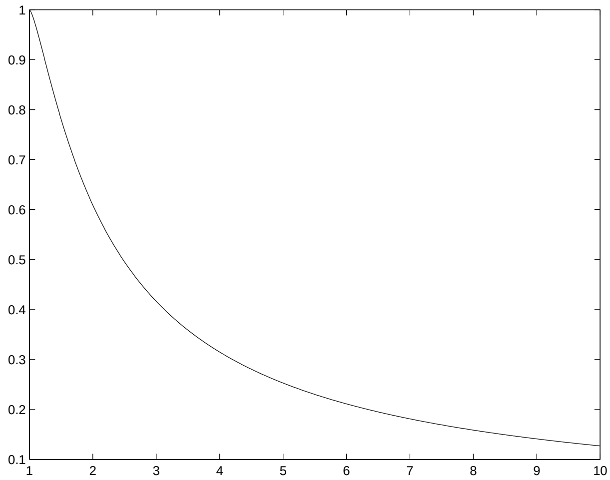


Note: The curve does not pass exactly through -5, which is the correct value according to the analytical calculation, and this is caused by the automaticaly selection of frequency points in the Matlab function.

The describing function is real and can hence be plotted according to

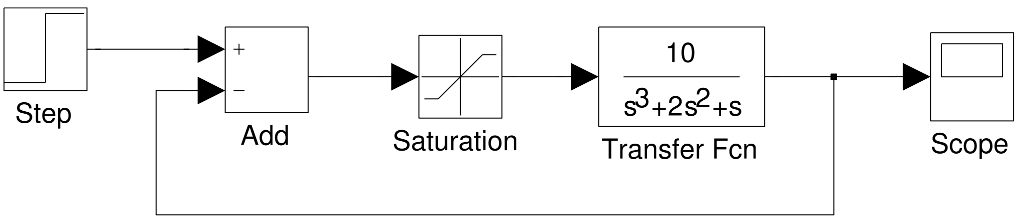
```
>> C=1:0.01:10;
>> Yf=2/pi*(asin(1./C)+1./C.*sqrt(1-C.^(-2)));
>> plot(C,Yf)
```

which gives



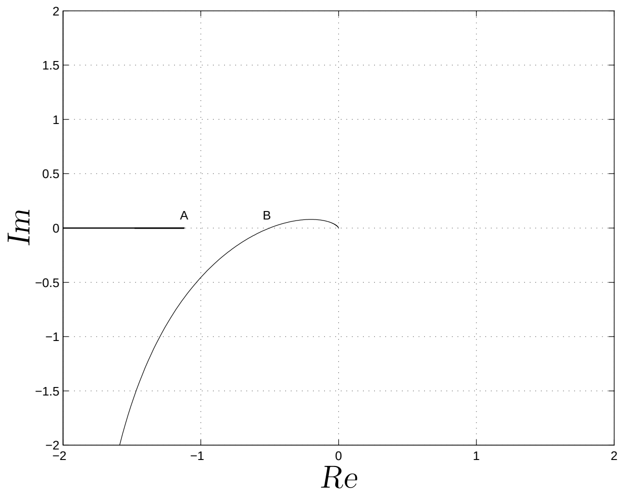
By zooming in one find that $Y_f = 0.2$ for $C = 6.35$

(b) An example of a Simulink model of the control system is shown in the figure below. A step with small amplitude is sufficient to start the oscillations. The simulation results agree very well with the theoretical values from a).



14.2

The Nyquist curve and the describing function are plotted below



The describing function changes direction in a point A, which corresponds to when $Y_f(C)$ takes it maximum value. The corresponding value of C can be found by differentiating $Y_f(C)$ with respect to C. Differentiation of

$$Y_f(C) = \frac{4H}{\pi C}\sqrt{1 - D^2/C^2}$$

with respect to C gives that the derivative is zero for $C = \sqrt{2}D$ and that $A = -\frac{\pi D}{2H}$. A possible intersection occurs when the Nyquist curve crosses the negative real axis. We have that $\arg G(i\omega) = -\pi$ when $\omega = 1$, and $|G(i1)| = 1/2$. The point B is thus $-1/2$. That the oscillation barely can exist means that $B \approx A$. The amplitude of the oscillation is 2.5 yields $\sqrt{2}D = 2.5$. Hence, $D = 5 \cdot \sqrt{2}/4$ and $H = \pi \cdot 5 \cdot \sqrt{2}/4$. The frequency of the oscillation is $\omega = 1$.

Go back

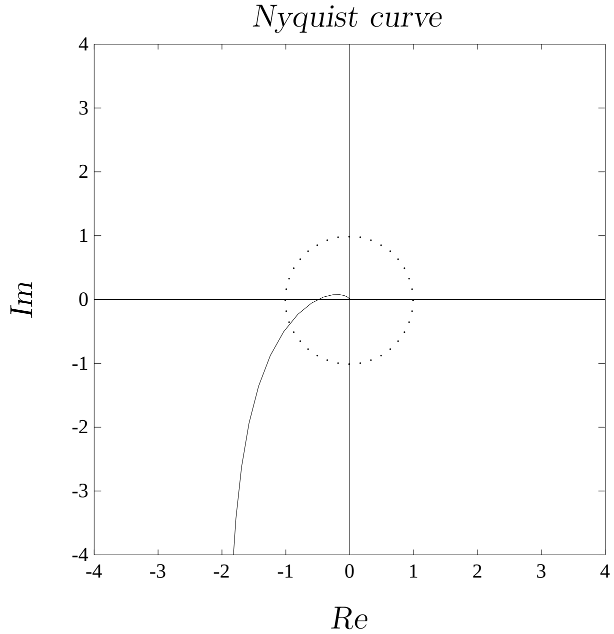
14.3

(a) The describing function of the relay is

$$Y_f(C) = 4/(\pi C), \quad \Rightarrow \quad -1/Y_f(C) = -\pi C/4$$

The curve $-1/Y_f(C)$ covers the entire negative real axis. The frequency response is

$$\begin{aligned}
G(i\omega) &= \frac{K}{i\omega(i\omega+1)^2} = \frac{K(1-i\omega)^2 \cdot (-i\omega)}{\omega^2(1+\omega^2)^2} \\
&= \frac{K(1-\omega^2-2i\omega)(-i\omega)}{\omega^2(1+\omega^2)^2} = \frac{-2K\omega - iK(1-\omega^2)}{\omega(1+\omega^2)^2} \\
\arg G(i\omega) &= \arg(K) - \arg(i\omega) - 2\arg(1+i\omega) \\
&= 0 - \pi/2 - 2\text{atan}(\omega)
\end{aligned}$$



We will always have an oscillation as the Nyquist curve intersects with $-1/Y_f(C)$ for all values of K.

(b) At the intersection point we have that $\arg G(i\omega) = -\pi$, or alternatively that the imaginary part is 0. This occurs for $\omega = 1$. As $|G(i1)| = K/2$ the amplitude of the oscillation is given by

$$-\frac{K}{2} = -\frac{\pi C}{4}$$

The requirement that $C < 0.1$ results in $K < \pi/20$.

(c) With a possibly dynamic feedback $L(s)$, the phase of the linear loop-gain will be $\arg L(i\omega)G(i\omega) = \arg L(i\omega) + \arg G(i\omega)$. A controller yielding a phase lead (positive phase, rotating the Nyquist curve counter-clockwise) at $\omega \geq 1$ will thus allow us to use an increased gain K. One such controller is a PD-controller $1 + T_D s$ which will have the phase $\text{atan}(T_D)$ at $\omega = 1$. Note that the phase of $L(i\omega)G(i\omega)$ now asymptotically tends to $-\pi$ instead of $-3\pi/2$ when $w \to \infty$, and for sufficiently large T_D the Nyquist curve does not even cross the real axis.

```
>> s = tf('s');
>> G = 1/(s*(1+s)^2);
>> L1 = 1;L2 = 1 + 0.1*s; L3 = 1 + 0.25*s;L4 = 1+2*s;
>> nyquist(L1*G,L2*G,L3*G,L4*G);
>> axis([-1 0 -1 1]);
>> figure
>> bode(L1*G,L2*G,L3*G,L4*G);
```

Go back

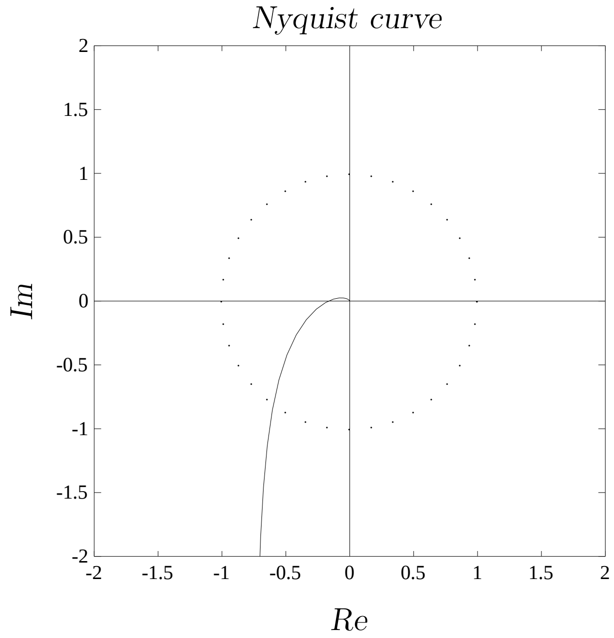
14.4

The describing function of an ideal relay:

$$Y_f(C) = \frac{4}{\pi \cdot C} \quad \Rightarrow \quad -\frac{1}{Y_f(C)} = -\frac{\pi}{4} \cdot C$$

(a) Plot the Nyquist curve of $G(s)H(s) = G(s)$

$$\begin{aligned} G(i\omega) &= \frac{1}{i\omega(i\omega+1)(i\omega+2)} = \frac{-i(1-i\omega)(2-i\omega)}{\omega(\omega^2+1)(\omega^2+4)} \\ &= -\frac{3}{(\omega^2+1)(\omega^2+4)} - i\frac{2-\omega^2}{\omega(\omega^2+1)(\omega^2+4)} \end{aligned}$$



If the point $-1/Y_N(C)$ is encircled by the Nyquist curve the amplitude of the oscillation will increase and otherwise it will decrease. This results in a stable oscillation. The frequency and amplitude can be determined from the intersection of the curves which occurs when $\text{Im}\, G(i\omega) = 0$, i.e. when $\omega = \sqrt{2}$. As $\text{Re}\, G(i\sqrt{2}) = -1/6$, we get

$$-1/6 = -\frac{\pi C}{4} \quad \Rightarrow \quad C = \frac{2}{3\pi}$$

Hence, the oscillation has the amplitude $2/(3\pi)$ and the frequency $\omega = \sqrt{2}$.

(b) Study $G(i\omega)H(i\omega)$

$$\begin{aligned} G(i\omega) \cdot H(i\omega) &= \frac{-i(1-i\omega)(2-i\omega)(1+Ki\omega)}{\omega(\omega^2+1)(\omega^2+4)} \\ &= \frac{-3+2K-K\omega^2}{(\omega^2+1)(\omega^2+4)} + i\frac{-2+\omega^2-3K\omega^2}{\omega(\omega^2+1)(\omega^2+4)} \end{aligned}$$

According to (a), we will avoid oscillations if $\text{Im}\, G(i\omega)H(i\omega) < 0, \ \forall \omega$.

$$\begin{aligned} &-2+\omega^2-3K\omega^2 < 0 \quad \Rightarrow \\ &K > \frac{\omega^2-2}{3\omega^2} \end{aligned}$$

As $(\omega^2-2)/(3\omega^2) < 1/3, \ \forall \omega$ we can choose any $K > 1/3$.

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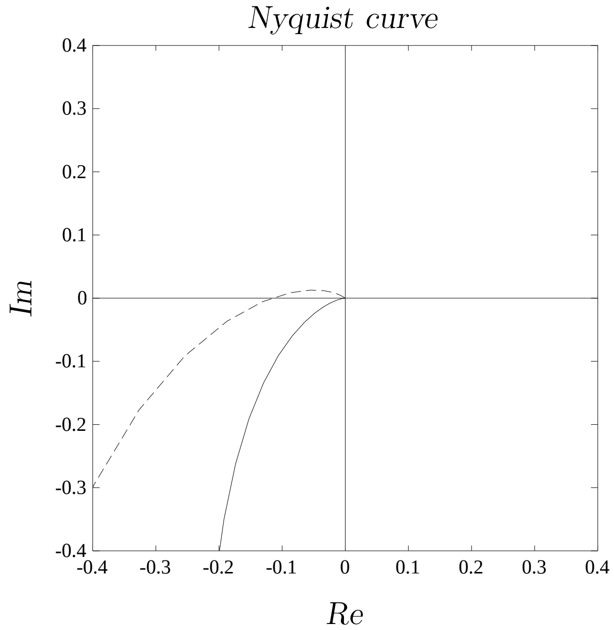


14.5

(a) Alternative (i): The describing function of a relay with hysteresis is given by

$$\begin{aligned} Y_f(C) &= \frac{4}{\pi C}\left(\sqrt{1-1/(2C)^2} - i/(2C)\right), \quad C \geq 0.5 \\ -1/Y_f(C) &= -\frac{\pi C}{4}\sqrt{1-1/(2C)^2} - i\frac{\pi}{8} \end{aligned}$$

which means that the imaginary part of $-1/Y_f$ will be $-\pi/8$ independent of C and the real part will start at zero and tend to $-\infty$.

The transfer function of the linear part is

$$G(s) = \frac{1}{s(s+1)}$$

Since the transfer function contains an integrator the argument will start at $-90°$ and since the relative degree is two the argument will tend to $-180°$. This indicates that there will be an intersection between $G(i\omega)$ and $-1/Y_f$. From the transfer function we have

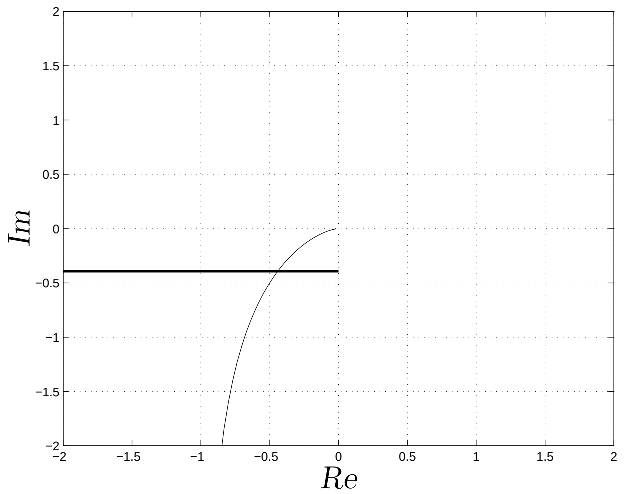
$$G(i\omega) = \frac{1}{i\omega(1+i\omega)} = \frac{-w-i}{\omega(1+\omega^2)}$$

Putting the real and imaginary parts of $G(i\omega)$ and $-1/Y_f$ equal to each other gives

$$\begin{aligned} \frac{1}{\omega(1+\omega^2)} &= \frac{\pi}{8} \\ \frac{\pi C}{4}\sqrt{1-1/(2C)^2} &= \frac{1}{1+\omega^2} \end{aligned}$$

The first equation has the approximate solution $\omega = 1.125$, which inserted in the second equation implies the solution $C = 0.75$.

Plot the Nyquist curve and the describing function.



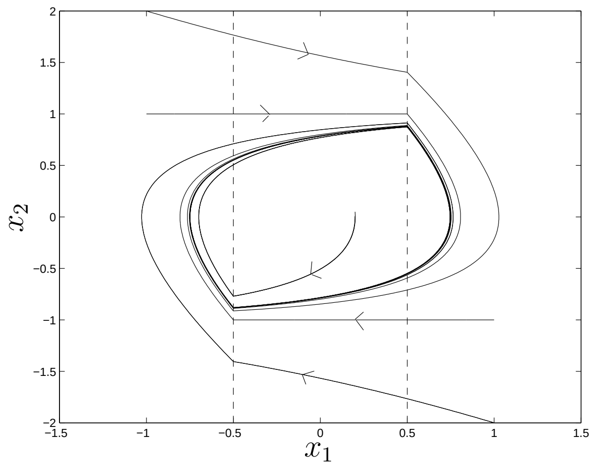
The curves intersect when $\omega = 1.235, C = 0.75$. This result can be found by looking at the plot or by solving the system of equations

C small $\Rightarrow$ $-1/Y_f(C)$ is encircled $\Rightarrow$ the amplitude of the oscillation increases
C large $\Rightarrow$ $-1/Y_f(C)$ is not encircled $\Rightarrow$ the amplitude of the oscillation decreases.
Thus, the oscillation is stable.

(b) Build a model in Simulink and verify the result.

(c) $x_1 = \theta,\ x_2 = \dot{\theta}$ yield

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 + u\end{aligned}\ , \qquad u = \begin{cases} 1, & x_1 < -0.5 \\ -1, & x_1 > 0.5 \end{cases}$$



Go back

14.6

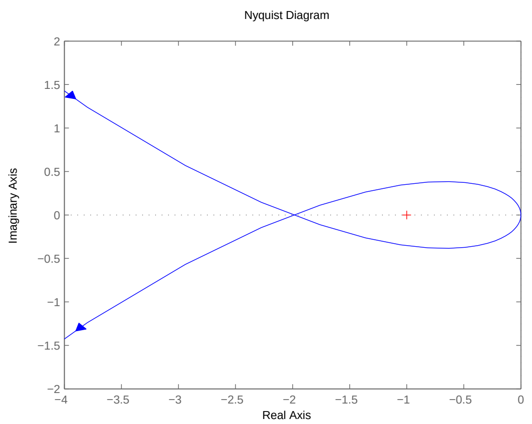
Inserting the numerical values for the PID coefficients gives the transfer function

$$G(s) = \frac{s^2 + 2s + 1}{s^3}$$

for the controller together with the motor. Evaluating G for $s = i\omega$ gives

$$G(i\omega) = \frac{-2\omega + i(1 - \omega^2)}{\omega^3}$$

It follows that G crosses the negative real axis at $\omega = \pm 1$ with $G(i) = -2$. A plot of the Nyquist curve is given below.

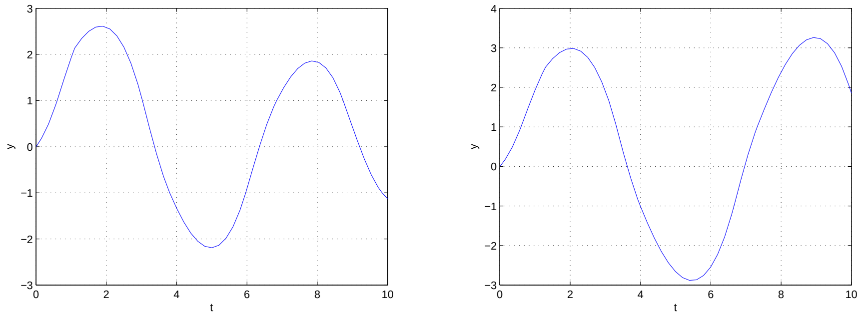


(a) Since the point -1 is not encircled by the Nyquist curve the closed loop system is asymptoticaldly stable when the amplifier is linear.

(b) The describing function for the saturation is

$$Y_f = \frac{2}{\pi}\left(\arcsin\frac{1}{C} + \frac{1}{C}\sqrt{1 - \frac{1}{C^2}}\right)$$

The condition $GY_f = -1$ gives $Y_f = 0.5$ which in turn gives $C \approx 2.5$. For values of C less than ≈ 2.5 the point $-1/Y_f(C)$ is not encircled so the amplitude ought to decrease, while for values of C greater than ≈ 2.5 the point $-1/Y_f(C)$ is encircled which indicates an increasing amplitude. The oscillation with $\omega = 1$ and $C \approx 2.5$ therefore probably has un unstable amplitude. This is confirmed by simulation. Below the output of the linear part is plotted for different initial amplitudes, showing a decreasing and an increasing oscillation.



It is clear that the control system will work well as long as there is no disturbance large enough to start an oscillation with an amplitude above the critical limit. (The growing oscillations that are created by large disturbances can be seen as a *windup* phenomenon of the integrator part of the regulator. When controlling a double integrator using a PID controller it is therefore very important to have some form of *anti-windup* compensation of the integral part.)

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14.7

The describing function is real. The Bode diagram shows that $\arg G_O(i\omega) = -180^\circ$ and $\mid G_O(i\omega) \mid = 2$ at $\omega = 2$. This implies that the Nyquist curve crosses the negative real axis in the point -2 for $\omega = 2$. We hence have to solve the equation

$$-2 = \frac{-1}{Y_f(C)}$$

which implies

$$Y_f(C) = \frac{4}{\pi C}\sqrt{1 - \frac{1}{C^2}} = \frac{1}{2}$$

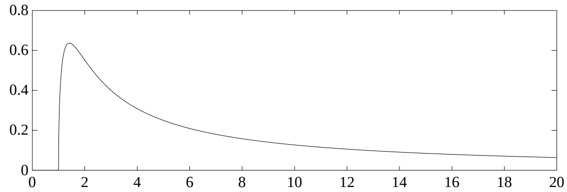
and

$$C^4 - \frac{64}{\pi^2}C^2 + \frac{64}{\pi^2} = 0$$

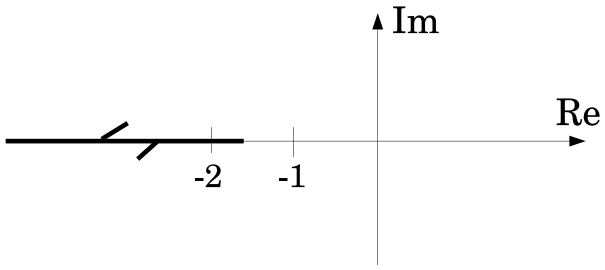
This gives

$$C = \overset{+}{(-)}\, 2.29 \quad \text{resp} \quad \overset{+}{(-)}\, 1.11$$

By inserting some values of C one realizes that $Y_f(C)$ looks like the figure

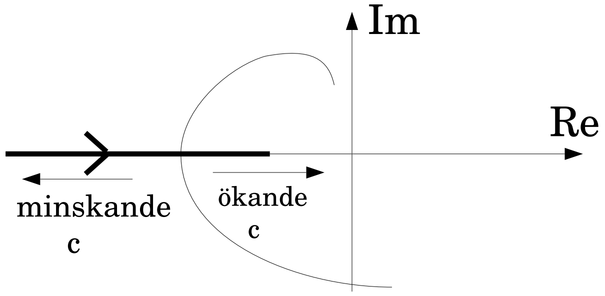


The function $-1/Y_f(C)$ thus moves along the real axis from $-\infty$ towards 0 when C increases, but stops at rougly $-1/.6$ and starts moving back towards $-\infty$. Hence, the curve $-1/Y_f(C)$ will intersect the Nyquist curve twice, as the computations indicate.



Analysis of the two candidate solutions gives

(I). C=1.11. For fixed amplitudes smaller than this value, when we think of the nonlinearity as a static gain with gain $Y_f(C)$, the point $-1/Y_f(C)$ will act as the point -1 in linear stability analysis, and tells us that the closed-loop system in a linear analysis would be asymptotically stable since it is not encircled. That means that any oscillation would decay, and C would not be constant as assumed. Instead it must decrease, and a new thought fixed value of C would once again indicate asymptotic stability. Hence, if initial oscillations are small, we suspect they will die out. (The relay here has a dead-zone which zeroes out everything between -1 and 1 so the result is reasonable, as a sinusoidal with amplitude 1.1 will almost completely be zeroed out and almost no energy enters the system. If the open-loop system G_0 is stable it is reasonable that the output will go to zero if the input almost always is zero)



(II). C=2.29. For fixed amplitudes larger than this value, when we think of the nonlinearity as a static gain with gain $Y_f(C)$, the point $-1/Y_f(C)$ will act as the point -1 in linear stability analysis, and tells us that the closed-loop system in a linear analysis would be asymptotically stable. That means that any oscillation would decay, and C would not be constant as assumed. Instead it would decrease, and a new thought fixed value of C would once again indicate asymptotic stability and C would decrease. However, if it decreases below 2.29, the point $-1/Y_f(C)$ is encircled by the Nyquist curve, and linear analysis tells us the system would be unstable and C would have to increase. At 2.29, we reach a stationary case were we neither increase nor decrease C according to linear theory, and we should suspect we will have oscillations with this amplitude. The limit cycle will have amplitude $C = 2.29$ and angular frequency $\omega = 2$.

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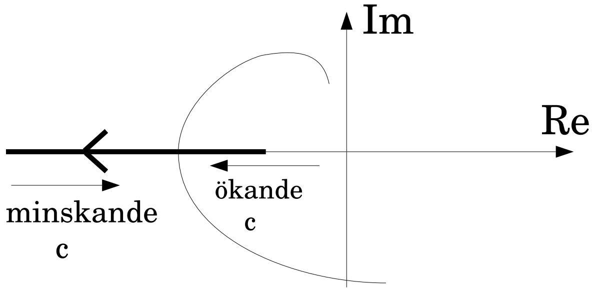


17 To Compensate Exactly for Nonlinearities

17.1

If we let

$$u = r - \cos x_1$$

we get a linear closed-loop system.

Go back

17.2

The control signal

$$u = -y^4 + y^2 + r = -x_1^4 + x_1^2 + r$$

results in an exact feedback linearization.

Go back

17.3

The system is defined by

$$(*) \qquad \begin{cases} \dot{x}_1 = x_1^2 + x_2 \\ \dot{x}_2 = u \\ y = x_1 \end{cases}$$

Make the change of variables:

$$\begin{cases} z_1 = y \\ z_2 = \dot{y} \end{cases}$$

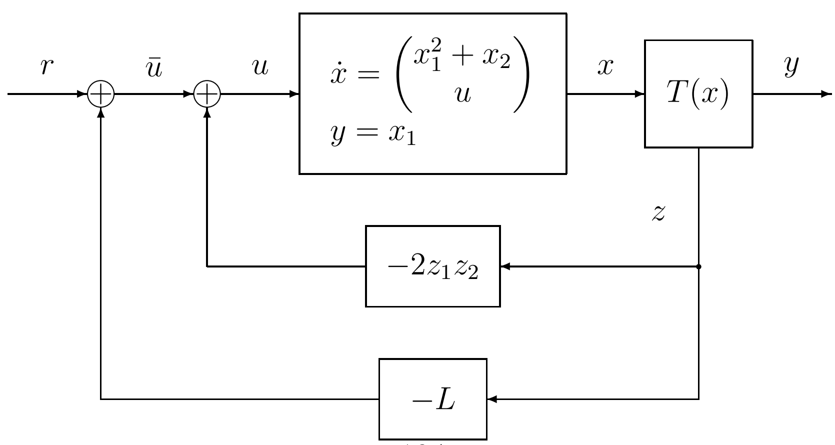
This results in

$$\dot{z}_1 = \dot{y} = z_2$$

$$\begin{aligned} \dot{z}_2 &= \ddot{y} = \frac{d}{dt}(\dot{x}_1) \\ &= \frac{d}{dt}\left(x_1^2 + x_2\right) \\ &= 2x_1\dot{x}_1 + \dot{x}_2 = [\text{according to } (*)] \\ &= 2x_1\left(x_1^2 + x_2\right) + u \\ &= \begin{bmatrix} (*) \Rightarrow x_2 = \dot{x}_1 - x_1^2 \\ x_1 = y = z_1 \\ \dot{x}_1 = \dot{y} = z_2 \end{bmatrix} \\ &= 2z_1\left(z_1^2 + z_2 - z_1^2\right) + u \\ &= 2z_1z_2 + u = \alpha(z) + \beta(z)u \end{aligned}$$

An exact feedback linearization results from

$$u = \frac{-\alpha(z) + \bar{u}}{\beta(z)} = -2z_1z_2 + \bar{u}.$$



Go back

17.4

As $\dot{x}_1$ depends on u we cannot choose y to be x_1. Hence, choose $y = x_2$.

$$\begin{aligned}
\dot{y} &= \dot{x}_2 = \sqrt{1+x_1} - \sqrt{1+x_2} \\
\ddot{y} &= \ddot{x}_2 = \frac{d}{dt}\left(\sqrt{1+x_1} - \sqrt{1+x_2}\right) = \\
&= \frac{1}{2\sqrt{1+x_1}}\dot{x}_1 - \frac{1}{2\sqrt{1+x_2}}\dot{x}_2 = \cdots = \\
&= \frac{1}{2}\left(\frac{1}{\sqrt{1+x_1}} - \frac{\sqrt{1+x_1}}{\sqrt{1+x_2}}\right) + \frac{u}{2\sqrt{1+x_1}}
\end{aligned}$$

Thus, the relative degree is 2. Now, do the change of variables $z_1 = y, z_2 = \dot{y} \Rightarrow$

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= /\text{from above}/ = \frac{1}{2}\left(\frac{1}{\sqrt{1+x_1}} - \frac{\sqrt{1+x_1}}{\sqrt{1+x_2}}\right) + \frac{u}{2\sqrt{1+x_1}} = \\
&= \frac{1}{2}\left(\frac{1}{z_2+\sqrt{1+z_1}} - \frac{z_2+\sqrt{1+z_1}}{\sqrt{1+z_1}}\right) + \frac{1}{2}\frac{1}{z_2+\sqrt{1+z_1}}u = \\
&= \alpha(z) + \beta(z)u
\end{aligned}$$

Choose $u = \frac{1}{\beta(z)}\left(\bar{u} - \alpha(z)\right)$ to get an exact feedback linearization. What are the poles of the system?

Go back

17.5

(a) The force is

$$m\ddot{y} = F - k(y) - d(\dot{y})$$

Do the change of variables $x_1 = y$, $x_2 = \dot{y}$ and $x_3 = F$ which results in the state-space form

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{1}{m}(-k(x_1) - d(x_2) + x_3) \\
\dot{x}_3 &= -x_3 + u \\
y &= x_1
\end{aligned}$$

(b) Relative degree ν? Differentiate y with respect to time

$$\begin{aligned}
\dot{y} &= \dot{x}_1 = x_2 \\
\ddot{y} &= \dot{x}_2 = \frac{1}{m}(-k(x_1) - d(x_2) + x_3) \\
y^{(3)} &= \frac{1}{m}(-k'(x_1)\dot{x}_1 - d'(x_2)\dot{x}_2 + \dot{x}_3) \\
&= \frac{1}{m}(-k'(x_1)\dot{x}_1 - d'(x_2)\dot{x}_2 - x_3 + u)
\end{aligned}$$

As $\nu = n = 3$ we can make an exact feedback linearization. Make the change of variables

$$\begin{cases} z_1 = y \\ z_2 = \dot{y} \\ z_3 = \ddot{y} \end{cases} \quad \Leftrightarrow \quad \begin{cases} x_1 = z_1 \\ x_2 = z_2 \\ x_3 = k(z_1) + d(z_2) + mz_3 \end{cases}$$

which results in the state-space form

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
\dot{z}_3 &= \frac{1}{m}(-k'(z_1)z_2 - d'(z_2)z_3 - k(z_1) - d(z_2) - mz_3 + u) \\
y &= z_1
\end{aligned}$$

The control signal

$$u = m\tilde{u} + k'(z_1)z_2 + d'(z_2)z_3 + k(z_1) + d(z_2) + mz_3$$

results in a linear system from $\tilde{u}$ y.

Go back